Connectivity augmentation in planar straight line graphs*

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Abstract

It is shown that every connected planar straight line graph with $n \geq 3$ vertices has an embedding preserving augmentation to a 2-edge connected planar straight line graph with at most $\lfloor (2n-2)/3 \rfloor$ new edges. It is also shown that every planar straight line tree with $n \geq 3$ vertices has an embedding preserving augmentation to a 2-edge connected planar topological graph with at most $\lfloor n/2 \rfloor$ new edges. These bounds are best possible. However, for every $n \geq 3$, there are planar straight line trees with n vertices that do not have an embedding preserving augmentation to a 2-edge connected planar straight line graph with fewer than $\frac{17}{33}n - O(1)$ new edges.

1 Introduction

Vertex- and *edge-connectivity augmentation* are important optimization problems in network design. Given an undirected graph $G = (V, E)$ and an integer k, they ask for the minimum augmenting edge set F such that the graph $G' = (V, E \cup F)$ is k-connected or k-edge connected, respectively. Eswaran and Tarjan [3, 26] and Plesník [24] showed independently that both problems can be solved in linear time for $k = 2$. Jackson and Jordán [13] showed that the vertex-connectivity problem can be solved in polynomial time for every fixed $k \in \mathbb{N}$. Végh [35] recently gave a polynomial time algorithm for augmenting the vertex-connectivity of a graph from $k-1$ to k if k is part of the input. For the edge-connectivity problem, Watanabe and Nakamura [37] gave a polynomial-time solution for every fixed $k \in \mathbb{N}$. Later Frank [6] has found a unified approach based on the edge-splitting technique by Lovász [17] and Mader [18]. Nagamochi and Ibaraki [21] proposed an algorithm for the edge-connectivity problem that runs, for every fixed $k \in \mathbb{N}$, in $O(nm \log n + nm \log^2 n)$ time for an input graph with n vertices and m edges. Refer to a survey by Nagamochi and Ibaraki [22] for other variants of connectivity augmentation, including weighted and directed versions, and to a survey by Kortsarz and Nutov [16] for approximation results.

In the *planarity preserving* version of the vertex- and edge-connectivity augmentation problem, both the input graph G and the output graph G' have to be planar (Fig. 1ab). Kant and Bodlaender [15] showed that the planarity preserving vertex-connectivity augmentation problem is NP-complete already for $k = 2$, and they gave a 2-approximation algorithm that runs in $O(n \log n)$ time. (Gutwenger *et al.* [9] has recently pointed out an error in a $\frac{5}{3}$-approximation algorithms by Fialko and Mutzel [5]). Rutter and Wolff [28] showed that the planarity preserving edge-connectivity augmentation problem is also NP-complete. Linear time algorithms for the planarity preserving versions are known for the case that $k = 2$ and the input G is an outerplanar graph [14, 20]; and for the version of the problem where both the input G and the output G' are required to be outerplanar [8].

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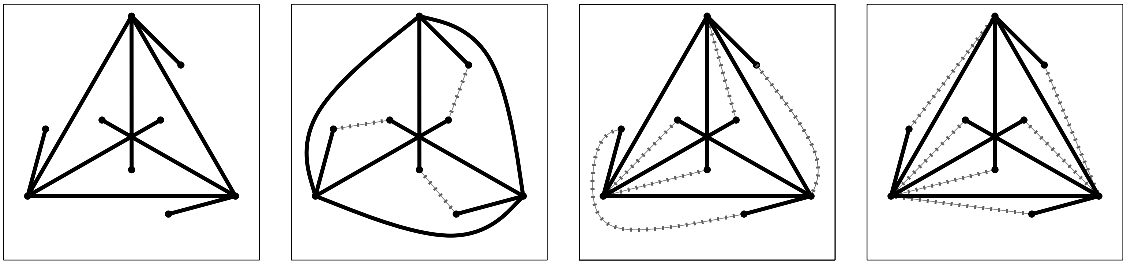


(a) (b) (c) (d)

Figure 1: (a) A PSLG. (b) A planarity preserving augmentation to a 2-edge connected planar graph with 3 new edges; (c) An embedding preserving augmentation to a 2-edge connected PTG with 5 new edges; and (d) an embedding preserving augmentation to a 2-edge connected PSLG with 6 new edges.

Sometimes it is not enough to preserve the planarity of a graph, but one would like to preserve the given planar embedding as well. A *planar topological graph* (PTG) is a simple planar graph together with an embedding in the plane, where the vertices are mapped to distinct points in the plane, every edge is mapped to a continuous arc between its endpoints, and the embeddings of any two edges are either disjoint or intersect at a common endpoint. A *planar straight line graph* (PSLG) is a planar topological graph where every edge is embedded in the plane as a straight line segment. By Fáry's Theorem [4, 36], every planar graph has an embedding in the plane as a PSLG. In the *embedding preserving* connectivity augmentation problems, we are given a PTG $G = (V, E)$ and an integer k, and we need to find the minimum augmenting edge set F such that $G' = (V, E \cup F)$ is a k-connected (resp., k-edge connected) PTG, and all edges in E have the same embedding in the input and the output graphs (Fig. 1c-d). Rappaport [27] proved that it is NP-hard to find the minimum number of edges necessary for an embedding preserving augmentation of a PSLG to a 2-edge connected PSLG. Rutter and Wolff [28] proved that this problem is already NP-hard for planar straight line *trees*. Gutwenger *et al.* [10] gave a near-linear time algorithm for the embedding preserving 2-connectivity augmentation problem on connected PTGs, but showed that the problem is NP-hard for disconnected PTGs.

Abellanas *et al.* [1] addressed combinatorial problems about the embedding preserving connectivity augmentation of certain types of PSLGs. They proved that every connected PSLG with n vertices has an embedding preserving augmentation to a 2-connected PSLG with at most $n-2$ new edges, and this bound is best possible. This is a strengthening of a previously known result that any (abstract) graph with n vertices can be augmented to a 2-connected graph by adding at most $n-2$ edges, which is best possible for a star with n vertices. The embedding of the input PSLG G, however, severely limits the possible new straight line edges. For edge-connectivity augmentation, Abellanas *et al.* [1] showed that every *planar straight line path* with n vertices has an embedding preserving augmentation to a 2-edge connected PSLG with at most $\lfloor n/2 \rfloor$ new edges, which is best possible for a zig-zag path on n points in convex position. In contrast, if the embedding of the input graph does not have to be preserved, or if the new edges do not have to be embedded as straight line segments, then a single new edge is enough to augment a path to cycle, which is 2-edge connected and planar (if this new edge is drawn as a straight line segment, however, it may cross edges of the input PSLG).

Abellanas *et al.* [1] showed that every connected PSLG with $n \geq 3$ vertices in general position in the plane has an embedding preserving augmentation to a 2-edge connected PSLG with at most $\frac{6}{7}n$ new edges, and sometimes $\lfloor (2n-2)/3 \rfloor$ new edges are necessary. Their lower bound construction for $n \geq 7$ is composed of a triangulation on $m \geq 3$ vertices with a leaf added in each bounded face and three leaves added in the unbounded triangular face, lying in distinct segments of the circumscribed circle of the triangle. Since a triangulation on $m \geq 3$ vertices has $2m-5$ bounded faces, the resulting PSLG has $n = 3m-2$ vertices and each of the $2m-2 = (2n-2)/3$ leaves requires a new edge to raise the vertex degree to 2. For $3 \leq n \leq 6$, the star graph gives the same lower bound, since $\lfloor (2n-2)/3 \rfloor = \lfloor n/2 \rfloor$. Abellanas *et al.* conjectured that their lower bound is tight. This paper confirms their conjecture.

Theorem 1 *Every connected* PSLG *with $n \geq 3$ vertices in general position in the plane has an embedding preserving augmentation to a 2-edge connected* PSLG *with at most $\lfloor (2n-2)/3 \rfloor$ new edges. This bound is best possible.*

A similar (but simpler) argument can be used for the embedding preserving augmentation of a PTG to a 2-edge connected PTG. A lower bound construction for $n \geq 5$ is composed of a triangulation on $m \geq 3$ vertices with a leaf added in each (bounded or unbounded) face. A triangulation on $m \geq 3$ vertices has $2m-4$ faces, the resulting PSLG has $n = 3m-4$ vertices and each of the $2m-4 = (2n-4)/3$ leaves requires a new edge to raise the vertex degree to 2. We show below that this lower bound is tight for $n \geq 7$. (The lower bound of $\lfloor n/2 \rfloor$, given by a star, is better for $n = 3$, 4, and 6.)

Theorem 2 *Every connected* PTG *with $n \geq 7$ vertices has an embedding preserving augmentation to a 2-edge connected* PTG *with at most $\lfloor (2n-4)/3 \rfloor$ new edges. This bound is best possible.*

There are PSLGs that have no embedding preserving augmentation to a *3*-edge connected PSLG. For a set S of $n \geq 3$ points in convex position, a maximal PSLG is a triangulation of the convex hull, which has a vertex of degree 2. Hence there is no 3-edge connected PSLG with vertex set S. Recently, Tóth and Valtr [34] proved that a PSLG $G = (V, E)$ has an embedding preserving augmentation to a 3-edge connected PSLG if and only if there is no edge $e \in E$ such that e is a chord of the convex hull $\mathrm{ch}(V)$ and all vertices on one side of e lie on the convex hull. Al-Jubeh *et al.* [2] showed that if a PSLG with $n \geq 4$ vertices has an embedding preserving augmentation to 3-edge connected PSLG, then $2n-2$ new edges are always sufficient and sometimes necessary for the augmentation. There are 3-edge connected PSLGs that have no embedding preserving augmentation to a 4-edge connected PTG. For instance, one vertex in any straight line embedding of K_4 is incident to three triangular faces, and the degree of this vertex remains 3 in any embedding preserving augmentation.

Trees. Abellanas *et al.* [1] proved that every *planar straight line tree* with $n \geq 3$ vertices in general position in the plane has an embedding preserving augmentation to a 2-edge connected PSLG with at most $\frac{2}{3}n$ new edges. The example of a star graph with n vertices shows that $\lfloor n/2 \rfloor$ new edges are sometimes necessary (independently of the embedding). We show that this bound is tight if we drop the condition that the new edges have to be straight line segments, and obtain a 2-edge connected PTG.

Theorem 3

(i) *Every planar topological tree with $n \geq 3$ vertices has an embedding preserving augmentation to a 2-edge connected* PTG *with at most $\lfloor n/2 \rfloor$ new edges. This bound is best possible.*

(ii) *Every planar topological tree with $n \geq 3$ vertices and k leaves has an embedding preserving augmentation to a 2-edge connected* PTG *with at most $\lfloor 2k/3 \rfloor$ new edges. This bound is best possible for $k \leq \lfloor \frac{n}{2} + 1 \rfloor$.*

However, if we insist on adding straight line edges only, then more than $\lfloor n/2 \rfloor$ new edges may be necessary. We present a new lower bound construction.

Theorem 4 *For every $k \geq 1$, there is a planar straight line tree with $n = 33k - 20$ vertices in general position in the plane that has no embedding preserving augmentation to a 2-edge connected* PSLG *with fewer than $17k - 10 = \frac{17}{33}n + \frac{10}{33}$ new edges.*

Terminology. A finite *planar topological graph (*PTG*)* G decomposes the plane into connected components, which are the *faces* of the graph. G has a unique *unbounded face*, all its remaining faces are *bounded*. Let $V(G)$, $E(G)$, and $F(G)$, respectively, denote the set of vertices, edges, and faces of G. Every edge is adjacent to two (not necessarily different) faces. An edge adjacent to the same face on both sides is a bridge. The *2-edge block*s (for short *block*s) of G are the maximal 2-edge connected subgraphs of G (some of which may be singletons). A block of G is *terminal* if it is incident to exactly one bridge. A block adjacent to the outer face is called an *outer block*.

If G is connected, then the boundary of each face is also connected. In particular, every bounded face is simply connected, and the complement of the unbounded face is also simply connected.

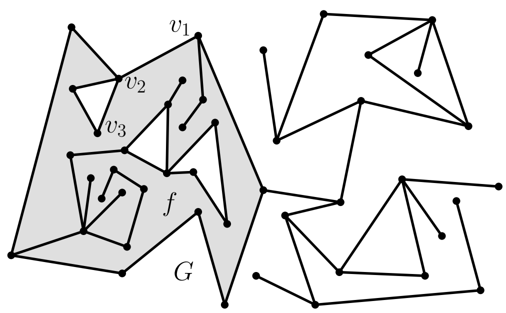


Figure 2: A PSLG G with 45 vertices, 24 blocks (20 of which are singletons), and 8 faces (including the outer face). Corner (v_1, v_2, v_3) is adjacent to face f. Vertex v_2 is the apex of two corners adjacent to f.

At every vertex $v \in V(G)$ of a PTG G, the incident edges have a circular order, called the *rotation of* v, which is the counterclockwise order in which they intersect any sufficiently small circle centered at v. A *corner* of a PTG G is a triple $c = (v_1, v_2, v_3)$ of vertices with $v_1v_2, v_2v_3 \in E(G)$, and the edges v_2v_1 and v_2v_3 are consecutive in the rotation of v. The *apex* of a corner $c = (v_1, v_2, v_3)$ is the vertex v_2, sometimes denoted $\hat{c}$. The corner $c = (v_1, v_2, v_3)$ is *adjacent* to a face $f \in G$ if f lies on the left side of both directed edges $\overrightarrow{v_1v_2}$ and $\overrightarrow{v_3v_2}$. Note that several corners at the same vertex may be adjacent to the same face, since the closure of a face is not necessarily simply connected.

For three distinct points in the plane, p_1, p_2, and p_3, the angular domain $\angle p_1p_2p_3$ is the intersection of the halfplanes lying on the right of $\overrightarrow{p_1p_2}$ and $\overrightarrow{p_2p_3}$. In a *planar straight line graph (*PSLG*)*, a corner $c = (v_1, v_2, v_3)$ is *convex* (respectively, *reflex*) if the open angular domain $\angle v_1v_2v_3$ is convex (resp., nonconvex). In particular, a vertex of degree 1 in a PSLG is incident to a unique corner of $360°$ angle, hence this corner is reflex. For a PSLG G, we denote by $\mathrm{ch}(G)$ the convex hull of the vertices of G. A face $f \in F(G)$ is an open set in the plane, and the closure of f is denoted by $\mathrm{cl}(f)$.

Removing double edges. A *planar topological (resp., straight line) multigraph* is a PTG (resp., PSLG) with a positive integral multiplicity assigned to every edge. It is *k-edge connected* for an integer k if and only if it is connected after deleting any subset of edges of total multiplicity at most $k - 1$. Abellanas *et al.* [1] proved an important lemma [1, Lemma 4] about transforming a 2-edge connected planar straight line multigraph into a 2-edge connected PSLG. This result generalizes to planar *topological* multigraphs with essentially the same proof. For completeness, we include the proof for PTGs.

Lemma 1 *Given a 2-edge connected planar topological multigraph (resp., planar straight line multigraph) G with $n \geq 3$ vertices and d, $d \in \mathbb{N}$, double edges, one can obtain a* simple *2-edge connected* PTG *(resp.,* PSLG*) by changing the multiplicity of every edge to 1 and adding at most d new edges (resp., d new straight line edges). Furthermore, if all bridges of G are adjacent to a face $f \in F(G)$, then all new edges lie in f.*

Proof. The proof for planar straight line multigraphs is available in [1]. Assume that G is a planar topological multigraph.

We proceed by induction on d. In the base case $d = 0$, graph G is already a 2-edge connected PTG. Assume that $d \geq 1$ and let e be an edge of multiplicity 2. By decreasing the multiplicity of e to 1, we obtain a planar topological multigraph G' having $d - 1$ double edges. If G' is 2-edge connected, then the induction step is complete. Otherwise, the only bridge of G' is e. Let G_1 and G_2 be the two connected components of $G' \setminus e$. Let $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$ be the endpoints of e. Let $f \in F(G')$ be the face adjacent to e (on both sides). Assume without loss of generality that G_1 has at least two vertices. Since G had no loops, there is an edge $e' \in E(G_1)$ incident to v_1 and adjacent to f. Let v_3 be the second endpoint of e'. Since both v_3 and v_2 are incident to face f, we can connect them by an arc in face f. Augment G' with this edge v_2v_3, which is an edge between the two components of $G' \setminus e$. We obtain a 2-edge connected planar topological multigraph having $d - 1$ double edges, hence the induction step is complete. □

As a consequence, if a PTG (resp., PSLG) G has an embedding preserving augmentation to a 2-connected planar topological (resp., straight line) multigraph with m new edges (possibly doubling some existing edges), then G also has an embedding preserving augmentation to a 2-edge connected *simple* PTG (resp., PSLG) with at most m new edges.

Corollary 1 *A* PSLG *G with b bridges and $n \geq 3$ vertices in general position in the plane has an embedding preserving augmentation to a 2-edge connected* PSLG *with at most b new edges. In particular, if all bridges are adjacent to a face $f \in F(G)$, then all new edges lie in f.*

Organization. The key tools, based on a closed curve and a dual graph, are introduced in Section 2. We illustrate the use of these tools for embedding preserving augmentation in the special case that the vertices of a PSLG are in convex position (Section 3). The same tools are also used for proving Theorem 3, on the embedding preserving augmentation of a tree to a 2-edge connected PTG (Section 4). In this section, we also show that more edges may be necessary if we insist to obtain a 2-edge connected PSLG. We then apply our general tools for geodesic curves (Section 5). This allows formulating a key lemma about embedding preserving augmentation of PSLGs if all bridges are adjacent to a single face (Section 6). Applying this result for each face of a PSLG or a PTG we prove Theorems 1 and 2 (Section 7).

2 A Jordan curve visits all blocks or all terminal blocks

Closed curves. A *closed curve* is an immersion of the unit circle into the Euclidean plane, represented by a function $\gamma : \mathbb{S} \to \mathbb{R}^2$. Consider a PSLG G with $n \geq 3$ vertices and a closed curve γ. We say that γ *visits* a corner (v_1, v_2, v_3) of a PTG G if there is a point $p \in \mathbb{S}$ such that $\gamma(p) = v_2$ and a small neighborhood of p is mapped into the area on the right of the directed edges $\overrightarrow{p_1p_2}$ and $\overrightarrow{p_2p_3}$ (in PSLGs, this is the angular domain $\angle v_1v_2v_3$). A closed curve γ visits a vertex $v \in V(G)$ if it visits a corner incident to v. A closed curve γ is *compatible with G* if (i) γ is disjoint from the relative interior of any edge of G, (ii) every self-intersection of γ lies at a vertex of G (i.e., $\gamma(p) = \gamma(q)$ and $p \neq q$ implies that $\gamma(p) \in V(G)$), and (iii) γ contains a vertex of G if and only if it visits that vertex. It is immediate that a closed curve compatible with G lies in the closure of a face of G. We define a PTG $H(\gamma)$, whose vertices are the vertices of G visited by γ, and the edges are the portions of γ between consecutive vertices along γ.

A closed curve γ compatible with G decomposes a face of G into connected components, which we call the *cell*s of γ (see Fig. 3). Let C denote the cells adjacent to both γ and some edges of G. We define the *dual graph* $D(\gamma)$ of the cells in C: the nodes corresponds to the cells in C, two nodes are connected by an edge if and only if they are adjacent to a same bridge of G (from opposite sides). We allow loops but no multiple edges in the dual graph. In particular, if the same cell lies on both sides of a bridge of G, then the corresponding node of $D(\gamma)$ has a loop, however, $D(\gamma)$ does not have double edges even if several bridges of G are adjacent to the same two cells in C.

Jordan curves. A closed curve $\gamma : \mathbb{S} \to \mathbb{R}^2$ is a *Jordan curve* if γ is injective, that is, if γ is an embedding of the unit circle into the Euclidean plane. In particular, a Jordan curve compatible with a PTG G visits every vertex of G at most once.

Proposition 1 *If γ is a Jordan curve compatible with a connected* PTG *G, then each cell in C is adjacent to at most one edge of $H(\gamma)$.*

Proof. Assume that an edge e of $H(\gamma)$ connects vertices $v_1, v_2 \in V(G)$. If $v_1 = v_2$, then $H(\gamma)$ has a single edge (a loop), and so no cell can be adjacent to more than one edge. Assume $v_1 \neq v_2$ and, without loss of generality, e is the counterclockwise arc along γ from v_1 to v_2. Since v_1 and v_2 lie on the boundary of the same face $f \in F(G)$ and G is connected, there is a counterclockwise path $L \subset G$ from v_1 to v_2 along the boundary of f. A cell in C adjacent to e is bounded by $e \subset \gamma$ and the path L. Hence this cell cannot be adjacent to any other edge of $H(\gamma)$. □

Proposition 2 *If γ is a Jordan curve compatible with a* PTG *G and visits a corner in each block of G, then the dual graph $D(\gamma)$ is a forest.*

Proof. Construct a planar embedding of the dual graph $D(\gamma)$ as follows. For each cell $c \in C$, embed the corresponding node of the dual graph at a point $p(c)$ in the interior of c, and connect it to the midpoints of the adjacent bridges of G by pairwise continuous arcs that meet at $p(c)$ only (such arcs exists since cell c is connected). For every bridge adjacent to two cells, c_1 and c_2, the union of two arcs incident to the midpoint of the bridge is an edge between $p(c_1)$ and $p(c_2)$.

Suppose that the dual graph contains a circuit (possibly a loop). This circuit is embedded as a simple closed curve β in the plane. Every edge of G crossed by β is a bridge. Since β crosses at least one bridge, it separates at least two blocks of G from each other. Since the Jordan curves γ and β do not cross each other, at least one of the two blocks is disjoint from γ. This contradicts our assumption that γ visits each block of G, hence the dual graph has no circuits. □

Proposition 3 *If γ is a Jordan curve compatible with a* PTG *G and visits a corner in each terminal block of G then the dual graph $D(\gamma)$ is 3-colorable.*

Proof. Construct the same planar embedding of the dual graph $D(\gamma)$ as in the proof of Proposition 2. Recall that all faces are on one side of γ (interior or exterior). Augment the dual graph $D(\gamma)$ with a new node, embedded at a point $\hat{p}$ on the opposite side of γ (exterior or interior, respectively), and connect $\hat{p}$ to every point p_c lying in a cell $c \in C$ adjacent to γ. We obtain a PTG on the vertex set $\{p_c : c \in C\} \cup \{\hat{p}\}$. This graph is planar and so it is 4-colorable. Hence the subgraph generated by the nodes adjacent to $\hat{p}$ is 3-colorable. □

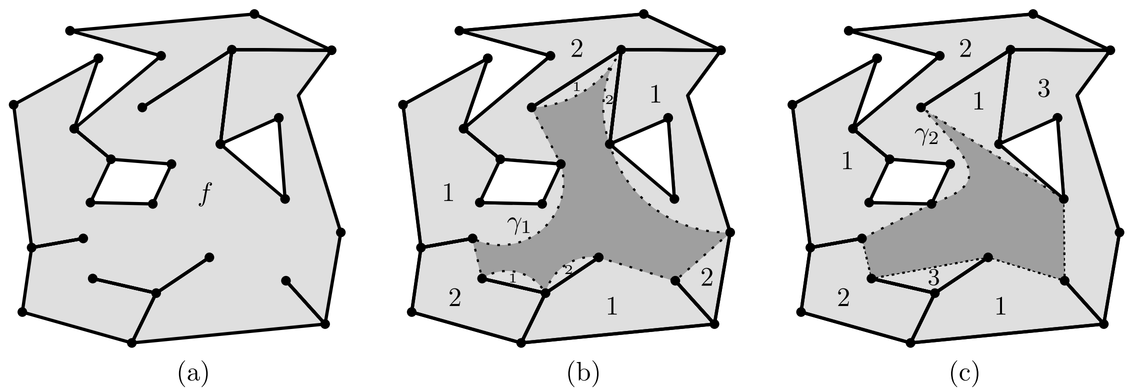


Figure 3: (a) a PSLG with all bridges adjacent to a single face f. (b) The Jordan curve γ_1 (dotted) that visits every block, and a 2-coloring of the cells induced by γ_1. (c) The Jordan curve γ_2 (dotted) that visits every terminal block, and a 3-coloring of the cells induced by γ_2.

Lemma 2 *Let G be a connected* PTG *with $n \geq 3$ vertices in general position in the plane. Let γ be a Jordan curve compatible with G that visits $m \geq 1$ vertices of some face f of G.*

(i) *If γ visits every block of G, then G has an embedding preserving augmentation to a 2-edge connected* PTG *with at most $\lfloor m/2 \rfloor$ new edges, each of which is an edge of $H(\gamma)$.*

(ii) *If, furthermore, G is a* PSLG *and every edge of $H(\gamma)$ is either a straight line segment or parallel to an edge of G, then the resulting 2-edge connected* PTG *is a* PSLG *and all new edges lie in f.*

Proof. By Proposition 2, the dual graph $D(\gamma)$ is a forest, and so it has a 2-coloring. See Fig. 3(a). By Proposition 1, each cell in C is adjacent to at most one edge of $H(\gamma)$. For each cell $c \in C$ in a smallest color class, augment G with the edge of $H(\gamma)$ adjacent to c. Together with the edges of f on the boundary of c, it forms a circuit (if c is a 2-gon, then the new edge is parallel to an edge of f). We have added at most $\lfloor m/2 \rfloor$ new edges. Every bridge of G as well as every new edge is now part of a circuit, and so the resulting planar topological multigraph is 2-edge connected. Lemma 1 completes the proof. □

Corollary 2 *Let G be a connected* PTG *with $n \geq 3$ vertices such that all bridges are adjacent to a face $f \in F(G)$. Let b denote the number of bridges of G. Then G has an embedding preserving augmentation to a 2-edge connected* PTG *with at most $\lceil b/2 \rceil$ new edges, all lying in f.*

Proof. Construct a Jordan curve γ compatible with G that visits one corner in each block of G. Since there are $b+1$ blocks, γ visits $b+1$ corners. By Lemma 2, G has an embedding preserving augmentation to a 2-edge connected PTG with at most $\lfloor (b+1)/2 \rfloor = \lceil b/2 \rceil$ new edges, all lying in f. □

Lemma 3 *Let G be a connected* PTG *with $n \geq 3$ vertices in general position in the plane. Let γ be a Jordan curve compatible with G that visits $m \geq 1$ corners of a face f of G.*

(i) *If γ visits every terminal block of G, then G has an embedding preserving augmentation to a 2-edge connected* PTG *with $\lfloor 2m/3 \rfloor$ new edges, each of which is an edge of $H(\gamma)$.*

(ii) *If, furthermore, G is a* PSLG *and every edge of $H(\gamma)$ is either a straight line segment or parallel to an edge of G, then the resulting 2-edge connected* PTG *is a* PSLG *and all new edges lie in face f.*

Proof. By Proposition 3, the dual graph of the cells in C has a 3-coloring. By Proposition 1, each cell in C is adjacent to at most one edge of $H(\gamma)$. For each cell $c \in C$ in the two smallest color classes, augment G with the edge of $H(\gamma)$ adjacent to c. Together with the edges of f on the boundary of c, it forms a circuit (if c is a 2-gon, then the new edge is parallel to an edge of f). We have added at most $m - \lceil m/3 \rceil = \lfloor 2m/3 \rfloor$ new edges. Every bridge of G as well as every new edge is now part of a circuit, and so the resulting planar topological multigraph is 2-edge connected. Lemma 1 completes the proof. □

3 Vertices in convex position

In this section we consider the special case of PSLGs whose vertices are in convex position in the plane. We prove a tight bound on the number of new edges necessary for the embedding preserving augmentation of a PSLG with b bridges and n vertices in convex position to a 2-edge connected PSLG. Note that Rutter and Wolff [29] recently gave an efficient algorithm for solving the embedding preserving 2-edge connectivity augmentation problem for any instance, in this special case.

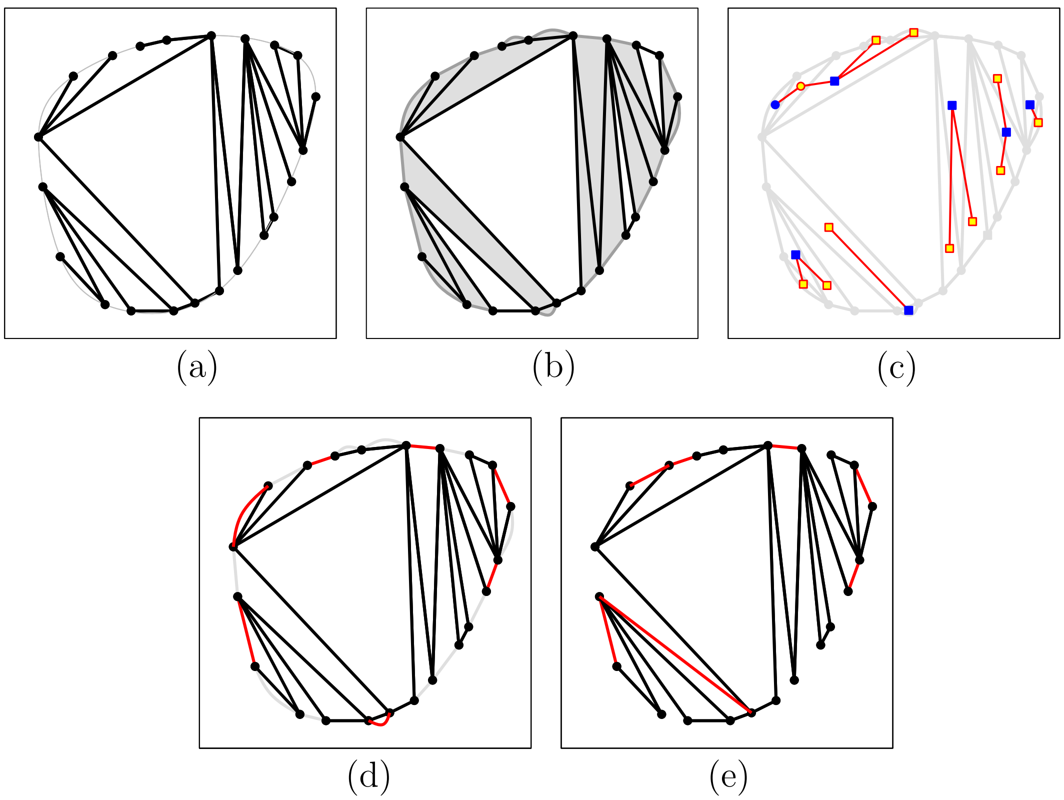


Figure 4: (a) A PSLG with vertices in convex position. (b) The edges of $\text{ch}(G)$ that connect distinct blocks. (c) A 2-coloring of the dual graph of C. (d) The resulting 2-edge connected planar straight line multigraph. (d) The resulting (simple) 2-edge connected planar straight line PSLG.

Theorem 5 *Let G be a connected PSLG with b bridges and $n \geq 3$ vertices in convex position in the plane. Then G has an embedding preserving augmentation to a 2-edge connected PSLG with at most $\min(b, \lfloor n/2 \rfloor)$ new edges, all lying in the outer face of G. This bound is best possible.*

Proof. We are given a PSLG G whose vertex set V is in convex position. Let γ be a closed Jordan curve compatible with G that visits the vertices of G in the order in which they appear along $\mathrm{ch}(G)$. By Lemma 2(ii), G has an embedding preserving augmentation to a 2-edge connected PSLG with at most $\lfloor n/2 \rfloor$ new straight line edges.

On the other hand, it is easy augment G to a 2-edge connected PSLG by adding b new straight line edges. By doubling every bridge, we obtain a 2-edge connected planar straight line multigraph. By Lemma 1, G has an embedding preserving augmentation to a 2-edge connected PSLG with b edges. The combination of the two upper bounds gives $\min(b\lfloor n/2 \rfloor)$.

Finally we present matching lower bound constructions. For all possible parameters $b, n \in \mathbb{N}$, with $3 \leq n$ and $0 \leq b \leq n-2$, we construct a PSLG with n vertices in convex position and with b bridges that cannot be augmented to a 2-edge connected PSLG with fewer than $\min(b, \lfloor n/2 \rfloor)$ new edges. Consider a circuit with $n-b$ vertices, embedded into the plane as a convex polygon inscribed in a circle. If $b \leq \lfloor n/2 \rfloor$, then add b leaves adjacent to distinct nodes of the cycle, and embed the leaves in distinct segments of the circle. The b leaves each require one additional edge to raise their degree to 2. If $\lfloor n/2 \rfloor < b$, then add a leaf in each of $n-b-1$ distinct segments of the circle, and add $2b-n+1$ leaves in the remaining one segment of the circle. The first $n-b-1$ leaves each require one new edge, and the remaining $2b-n+1$ leaves require $\lceil (2b-n+1)/2 \rceil = b+1-\lceil n/2 \rceil$ new edges. Altogether, at least $(n-b-1)+(b+1-\lceil n/2 \rceil) = \lfloor n/2 \rfloor$ new edges are necessary to obtain a 2-edge connected PSLG. □

4 Augmentation of planar straight line trees

In this section, we apply the techniques developed in Section 3, and prove that every planar topological tree with $n \geq 3$ vertices can be augmented to 2-edge connected PTG with at most $\lfloor n/2 \rfloor$ new edges (Theorem 3). However, if we insist on obtaining a 2-edge connected PSLG from a planar straight line tree with $n \geq 3$ vertices, then $\frac{17}{33}n + \frac{10}{33}$ new edges may be necessary (Theorem 4).

Proposition 4 *Let G be a planar topological tree, and let C be a subset of its corners. There is a Jordan curve γ compatible with G that visits exactly the corners in C.*

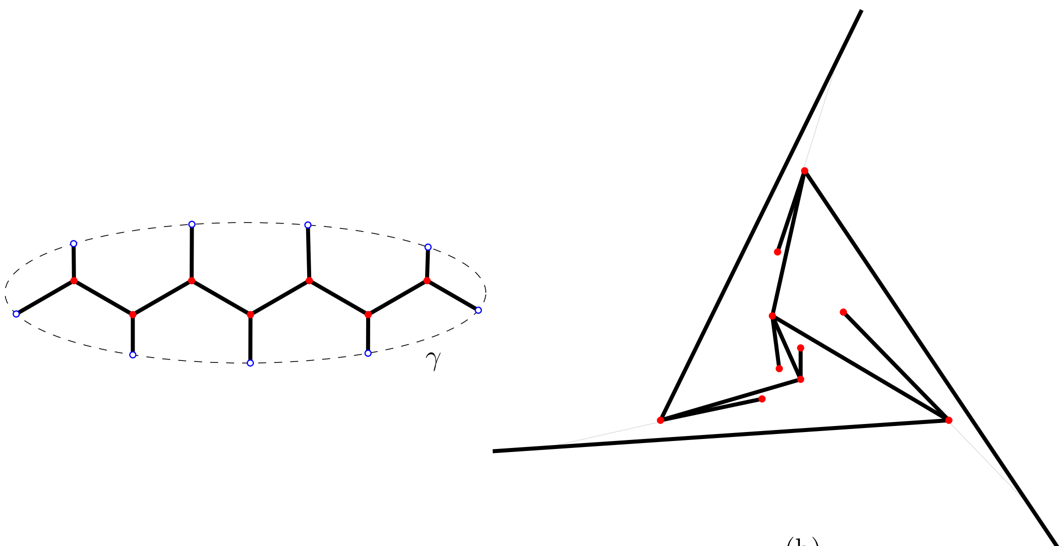
Proof. Let $\delta_0 > 0$ be a small constant such that the distance between any vertex and non-incident edge is at least $2\delta_0$. Then the set of points at distance δ, $0 \leq \delta < \delta_0$, from (the planar embedding of) G forms a Jordan curve $\gamma(\delta)$. For every $\varepsilon > 0$, there is a δ_ε, $0 < \delta_\varepsilon < \delta_0$, such that $\gamma(\delta_\varepsilon)$ intersects the disk of radius ε at every vertex of G. Let $\varepsilon > 0$ be so small such that the disk of radius ε centered at any vertex $v \in V(G)$ intersects only the edges incident to v. Modify the Jordan curve $\gamma(\delta_\varepsilon)$ in the ε-neighborhood of the apex of each corner in C to visit the corner. □

Proof of Theorem 3. Let G be a planar topological tree with $n \geq 3$ vertices. First we prove part (i) of Theorem 3. Let C be a subset of corners of G that consists of an arbitrary corner at each vertex of G. By Proposition 4, there is a Jordan curve γ compatible with G that visits every corner in C, hence it visits every vertex of G once. By Lemma 2, G has an embedding preserving augmentation to a 2-edge connected PTG with at most $\lfloor n/2 \rfloor$ new edges.

Next, we prove part (ii) of Theorem 3. Let C be a subset of corners of G that consists of of an arbitrary corner at each leaf of G. By Proposition 4, there is a Jordan curve γ compatible with G that visits every corner in C, hence it visits each of the k leaves of G once. By Lemma 2, G has an embedding preserving augmentation to a 2-edge connected PTG with at most $\lfloor 2k/3 \rfloor$ new edges. □

Proof of Theorem 4. For every integer $k \geq 1$, we construct a planar straight line tree $G(k)$ with $33k - 20$ vertices that cannot be augmented to a 2-edge connected PSLG with fewer than $17k-10$ new edges. Consider the section of a regular hexagonal tiling lying in a long and skinny ellipse γ depicted in Fig. 5a, including $3k - 2$ vertices of the tiling. The edges of the tiling clipped in the ellipse form the caterpillar graph $G_0(k)$, in which the leaves are the intersection points of γ and the edges of the tiling. Construct $G(k)$ from $G_0(k)$ by replacing each vertex of degree 3 with the construction in Fig. 5b, called *junction*. Each junction contains 10 vertices. Together with the $3k$ leaves along the ellipse, $G(k)$ has $n = 10(3k - 2) + 3k = 33k - 20$ vertices. We partition the vertex set of $G(k)$ into groups such that a group consists of either a vertex along γ or 10 vertices of a junction.

Observe that no two leaves within the same junction or within two different junctions can be connected by a new straight line edge. Furthermore, a leaf within a junction can only be connected to some other (nonleaf) vertex within the same junction.



(a) (b)

Figure 5: (a) A planar straight line tree with 7 vertices of degree 3, each adjacent to three convex corners; and with 9 leaves on the convex hull. (b) A junction consisting of 10 vertices: 5 leaves, and 5 matching reflex corners.

To augment a PSLG to a 2-edge connected PSLG, we must add new edges such that every bridge is contained in a circuit. By the above observation, 5 leaves in a junction require 5 new edges, and each of these new edges connects a leaf to a vertex in the same junction.

Next, consider the bridges of $G(k)$ between distinct groups of vertices. We show that at least $\lfloor 2m/3 \rfloor = 2k$ new edges are required to include these bridges in some circuits. The curve γ and $G(k)$ determine $3k$ cells and the dual graph $D(\gamma)$. Each junction is adjacent to three cells, which form a triangle in the dual graph. If we add fewer than $2k$ new edges between distinct groups of vertices, then there are three cells adjacent to a junction such that the three cells together contain at most one such new edge. Hence there are two adjacent cells that contain no new edge between distinct groups, and so the bridge on the common boundary of these cells is not included in any circuit. This shows that there must be at least $2k$ new edges connecting distinct junctions or leaves along the ellipse.

Altogether, we need at least $5(3k - 2) + 2k = 17k - 10$ new edges for an embedding preserving augmentation of $G(k)$ to a 2-edge connected PSLG. $\square$

5 Geodesic hulls of corners

The *geodesic hull* (also known as *relative convex hull*) was introduced by Sklansky *et al.* [30] and rediscovered by Toussaint [33]. It is a generalization of the convex hull for points lying in a simply connected domain. Recall that the convex hull of a point set S in the plane is the minimal set that contains S and is convex (that is, it contains the straight line segment between any two points in that set). Let D be a simply connected closed polygonal domain. For two points, $p_1, p_2 \in D$, denote by $\text{geod}_D(p_1, p_2)$ the shortest path between p_1 and p_2 that lies in D. For a finite point set $S \subset D$, the *geodesic hull* is the minimum set that contains S and also contains $\text{geod}_D(p_1, p_2)$ for any two points, p_1 and p_2, in that set.

We extend the definition of geodesic hulls to a set of corners adjacent to a face of a PSLG. The above definition cannot be used directly, since every face f is an *open* domain rather than a closed one, and the corners lie on the boundary of the face. In particular, a vertex can be the apex of several corners adjacent to the same face, and a face can lie on both sides of an edge. We use the concept of weakly simple polygons to approximate a face of a PSLG by a simple polygon.

Definition 1

- *A* polygon P *of size* $k \in \mathbb{N}$*, denoted* $P = (p_1, p_2, \ldots, p_k)$*, is a piecewise linear closed curve that passes through the points* p_i*,* $i = 1, 2, \ldots, k$*, in this cyclic order, and follows straight line segments between* p_i *and* $p_{i+1 \bmod k}$*. The points* p_i*,* $i = 1, 2, \ldots, k$*, are the vertices of* P*.*
- *A polygon* $P = (p_1, p_2, \ldots, p_k)$ *is* simple *if it is a Jordan curve (in particular, all vertices are distinct), and the interior of* P *lies on the right side of each edge* $p_i p_{i+1 \bmod k}$*.*
- *A polygon* $P = (p_1, p_2, \ldots, p_k)$ *is* weakly simple *if for any* $\varepsilon > 0$*, there is a point* p'_i *in the* ε*-neighborhood of each* p_i*,* $i = 1, 2, \ldots, k$*, such that* $P' = (p'_1, p'_2, \ldots, p'_k)$ *is a simple polygon.*

It is easy to see that for a PSLG G, the vertices and edges on the boundary of a face $f \in F(G)$ with k corners form a weakly simple polygon of size k. Every edge on the boundary of f participates in exactly two corners of f, we obtain a polygon $(p_1, p_2, \ldots, p_k)$ by concatenating the corners of f in counterclockwise order along the boundary of f. There is a one-to-one correspondence between the corners of f and the vertices p_i, $1 \le i \le k$. Assume that the vertices of G are in general position and let $\varepsilon_0 > 0$ be a small constant such that no line intersects three disks of radius $\varepsilon_0 > 0$ centered at vertices of f. Placing a point $p'_i = p'_i(\varepsilon)$ at distance $\min(\varepsilon_0, \varepsilon)$ from p_i on the bisector of the corresponding corner, we obtain a simple polygon $(p'_1, p'_2, \ldots, p'_k)$.

We next define a polygonal path and a weakly simple polygon compatible with a face of a PSLG. Recall that $\hat{c}$ denotes the apex of a corner c of a PSLG G.

Definition 2 *Let* G *be a* PSLG *and let* $f \in F(G)$ *be a face.*

- *A sequence* $(c_1, c_2, \ldots, c_k)$ *of corners of* f *defines a* polygonal path compatible with f *if* $(\hat{c}_1, \hat{c}_2, \ldots, \hat{c}_k)$ *is a polygonal path, and the line segment* $\hat{c}_i \hat{c}_{i+1}$ *lies in the angular domain of both* c_i *and* c_{i+1} *for* $i = 1, 2, \ldots, k-1$*.*
- *A cyclic sequence* $W = (c_1, c_2, \ldots, c_k)$ *of corners of* f *defines a* polygon compatible with f *if* $(\hat{c}_1, \hat{c}_2, \ldots, \hat{c}_k)$ *is a polygon, and the line segment* $\hat{c}_i \hat{c}_{i+1 \bmod k}$ *lies in the angular domain of both* c_i *and* $c_{i+1 \bmod k}$ *for* $i = 1, 2, \ldots, k$*.*

For example, the sequence W of all corners of f, in cyclic order along the boundary of f, is a weakly simple polygons compatible with f. We say that a weakly simple polygon $W = (c_1, c_2, \ldots, c_k)$ compatible with f *visits* the corners $c_1, c_2, \ldots, c_k$. In particular, it visits a corner m *times* if the corner appears m times in the sequence $(c_1, c_2, \ldots, c_k)$. The *length* of polygonal path (resp., polygon) $(p_1, p_2, \ldots, p_k)$ is the sum of the Euclidean lengths of its edges, $\sum_{i=1}^{k-1} \mathrm{dist}(p_i, p_{i+1})$ (resp., $\sum_{i=1}^{k} \mathrm{dist}(p_i, p_{i+1 \mod k})$). The *geodesic between the corners*, denoted $\mathrm{geod}(c_1, c_2)$, is the shortest polygonal path compatible with f between c_1 and c_2. We are now ready to define the geodesic hull of a set of corners in a face of a PSLG.

Definition 3 *Consider a* PSLG *G, a face $f \in F(G)$, and a set C of corners adjacent to f. The* geodesic hull *of the corners in C, denoted $\mathrm{gh}_f(C)$, is the shortest weakly simple polygon compatible with f such that*

- *$\mathrm{gh}_f(C)$ visits all corners in C; and*
- *if f is unbounded then G lies inside $\mathrm{gh}_f(C)$.*

Proposition 5 *Let G be a* PSLG*, $f \in F(G)$, and C be a set of corners adjacent to f.*

(i) *$\mathrm{gh}_f(C)$ visits every corner at most twice.*

(ii) *If $\mathrm{gh}_f(C)$ visits a corner c twice, then all edges of G and $\mathrm{gh}_f(C)$ incident to the apex $\hat{c}$ lie in a halfplane bounded by a line passing through $\hat{c}$ (in particular c is a reflex corner of G).*

(iii) *If $\mathrm{gh}_f(C)$ visits the corners in C only and $\mathrm{gh}_f(C)$ has a convex interior angle at $c \in C$, then $\mathrm{gh}_f(C \setminus \{c\})$ visits the corners in $C \setminus \{c\}$ only.*

Proof. (i) Assume that $\mathrm{gh}_f(C) = (c_1, c_2, \ldots, c_k)$ visits corner c_1 at least three times. We show that there is a weakly simple polygon W compatible with f that visits all corner in C, which is strictly shorter than $\mathrm{gh}_f(C)$. Assume that $c_1 = c_i = c_j$, and the polygonal chain $(c_1, c_2, \ldots, c_j)$ has exactly three vertices at c_1. Since $\mathrm{gh}_f(C)$ is a weakly simple polygon, all three angular domains $\angle \hat{c}_{i-1}\hat{c}_i\hat{c}_{i+1}$, $\angle \hat{c}_{j-1}\hat{c}_j\hat{c}_{j+1}$, and $\angle \hat{c}_k\hat{c}_1\hat{c}_2$ are in the exterior of $\mathrm{gh}_f(C)$, and so at least two of them has to be convex. Furthermore at most one of them contains edges of G incident to $\hat{c}_1$. Assume w.l.o.g. that $\angle \hat{c}_{i-1}\hat{c}_i\hat{c}_{i+1}$ is convex and contains no edges of G incident to $\hat{c}_1$. Replace the edges $\hat{c}_{i-1}\hat{c}_i$ and $\hat{c}_i\hat{c}_{i+1}$ of $\mathrm{gh}_f(C)$ with the geodesic $\mathrm{geod}(c_{i-1}, c_{i+1})$. See Fig. 6(a-b). The resulting weakly simple polygon W is compatible with f and strictly shorter than $\mathrm{gh}_f(C)$, a contradiction.

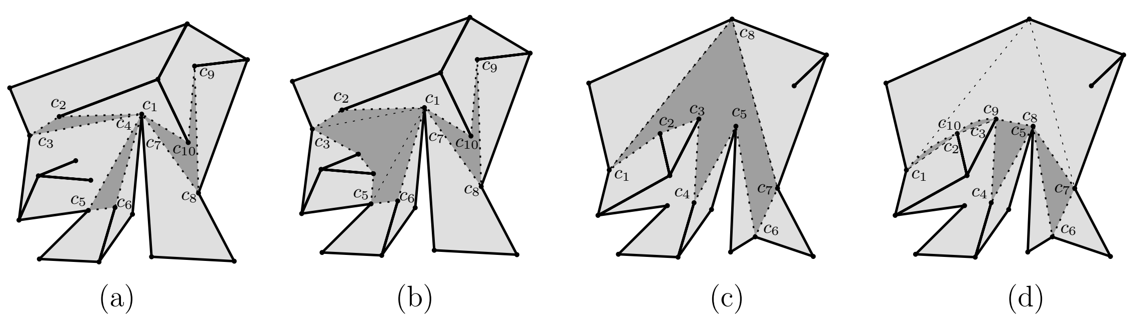


Figure 6: (a) A weakly simple polygon $(c_1, c_2, \ldots, c_{10})$ that visits c_1 three times. (b) Replacing path (c_3, c_4, c_5) with $\mathrm{geod}(c_3, c_4)$. (c) $\mathrm{gh}_f(C)$ has a convex interior angle at c_8. (d) Replacing the path $(c_7 c_8 c_1)$ with $\mathrm{geod}(c_7, c_7)$.

(ii) Assume that $\mathrm{gh}_f(C) = (c_1, c_2, \ldots, c_k)$ visits a corner twice, say $c_i = c_j$. If one of the angular domains $\angle \hat{c}_{i-1}\hat{c}_i\hat{c}_{i+1}$ and $\angle \hat{c}_{j-1}\hat{c}_j\hat{c}_{j+1}$ is convex and does not contain any edge of G incident to $\hat{c}_i$, then $\mathrm{gh}_f(C)$ is not a geodesic hull of C by the argument in part (i) above. So one of the two angular domains, say $\angle \hat{c}_{i-1}\hat{c}_i\hat{c}_{i+1}$, must be reflex and the other has to contain the edges of G incident to $\hat{c}_i$. A halfplane bounded by a supporting line of the reflex angle $\angle \hat{c}_{i-1}\hat{c}_i\hat{c}_{i+1}$ contains all edges of $\mathrm{gh}_f(C)$ incident to $\hat{c}_i$ as well as all edges of G incident to $\hat{c}_i$.

(iii) Assume that $\mathrm{gh}_f(C) = (c_1, c_2, \ldots, c_k)$ has a convex interior angle $\angle \hat{c}_{i+1}\hat{c}_i\hat{c}_{i+1}$ at corner c_i. We obtain $\mathrm{gh}_f(C \setminus \{c\})$ by replacing the edges $\hat{c}_{i-1}\hat{c}_i$ and $\hat{c}_i\hat{c}_{i+1}$ of $\mathrm{gh}_f(C)$ with the geodesic $\mathrm{geod}(c_{i-1}, c_{i+1})$. By the definition of geodesic hulls, all corners along $\mathrm{geod}(c_{i-1}, c_{i+1})$ are already contained in $\mathrm{gh}_f(C)$. See Fig. 6(c-d). Therefore, $\mathrm{gh}_f(C \setminus \{c\})$ does not visit c, and it visits corners in $C \setminus \{c\}$ only. □

6 All bridges along a single face

In this section, we consider a planar straight line multigraph G where all bridges are adjacent to a single face. We define a set of corners that span a geodesic hull visiting all blocks of G.

Definition 4 *Let G be a planar straight line multigraph such that all bridges are adjacent to a face f. A set C of corners adjacent to f is* full *if the following conditions are met:*

- *every terminal block is adjacent to a corner in C;*
- $\mathrm{gh}_f(C)$ *visits the corners in C only;*
- *if f is a bounded face, then C contains a convex corner adjacent to the outer block.*

It is clear that a set of *all* corners of a face f is full—we will use a *minimal* full set of corners. If f is bounded, a minimal full set of corners contains exactly one convex corner of the outer block by Proposition 5(iii). We call this special corner the *stem* corner in C.

Let C be a full set of corners. For the geodesic hull $\mathrm{gh}_f(C) = (p_1, p_2, \ldots, p_k)$, we construct a closed curve $\gamma(\mathrm{gh}_f(C))$ compatible with G as follows: If $\mathrm{gh}_f(C)$ visits only two corners (i.e., $\mathrm{gh}_f(C)$ consists of a double edge connecting two corners), then let $\gamma(\mathrm{gh}_f(C))$ be a Jordan curve compatible with G that visits these two vertices only. If $\mathrm{gh}_f(C)$ visits at least three corners, then construct $\gamma(\mathrm{gh}_f(C))$ from $\mathrm{gh}_f(C)$ by replacing every straight line edge $p_i p_{i+1 \bmod k}$ parallel to an edge of f with a circular arc lying in the sufficiently small neighborhood of the line segment $p_i p_{i+1 \bmod k}$ with the same endpoints. We define cells and the dual graph of $D(\mathrm{gh}_f(C)) := D(\gamma(\mathrm{gh}_f(C)))$ as in Section 3.

The dual graph $D(\mathrm{gh}_f(C))$ is not necessarily connected. We will process each component of $D(\mathrm{gh}_f(C))$ independently. We define a special subset of corners in C that plays the role of "separators" between these components. We show (Proposition 6) that if $\mathrm{gh}_f(C)$ is a simple polygon, then between any two distinct components of $D(\mathrm{gh}_f(C))$, the curve $\mathrm{gh}_f(C)$ has to visit a corner in the special set $A_f(C)$ defined below.

Definition 5 *For a full set C of corners of a face f, let $A_f(C) \subseteq C$ be the set of corners $c = (v_1, v_2, v_3)$ such that*

- *either*
  - $\mathrm{gh}_f(C)$ *has a reflex corner at c (hence c is a reflex corner of G), and*
  - *if edges v_1v_2, v_2v_3 are bridges, then the corresponding dual edges are in different components of $D(\mathrm{gh}_f(C))$;*
- *or c is the stem corner in C (in case f is bounded).*

The following proposition states the "separator" property of the corners in $A_f(C)$.

Proposition 6 *Let G be a planar straight line multigraph such that all bridges are adjacent to a face f. Let C be a minimal full set of corners such that $\mathrm{gh}_f(C)$ is a simple polygon. If two consecutive edges, e_1 and e_2, of $\mathrm{gh}_f(C)$ are adjacent to two distinct components of the dual graph $D(\mathrm{gh}_f(C))$, then the edges e_1 and e_2 meet at a corner in $A_f(C)$.*

Proof. Let c be the corner of f at which edges e_1 and e_2 meet. If f is bounded and c is the stem corner in C then our proof is complete. Assume now that $c = (v_1, v_2, v_3)$ is not the stem corner in C.

First we show that $\mathrm{gh}_f(C)$ has a reflex interior angle at c (and so c is also a reflex corner of C). Suppose, to the contrary, that $\mathrm{gh}_f(C)$ has a convex interior angle at c. By Proposition 5(iii), $\mathrm{gh}_f(C \setminus \{c\})$ does not visit c. We will show that $C \setminus \{c\}$ is also full, contradicting the minimality of C. If $C \setminus \{c\}$ is not full, then c is the unique corner in C adjacent to a terminal block. Assume that there is a terminal block in G such that the only adjacent corner in C is c. Then the cells adjacent to e_1 and e_2 are both adjacent to the unique bridge incident to this block. Hence, the two corresponding nodes in the dual graph $D(\mathrm{gh}_f(C))$ are adjacent. This, however, is impossible since we assumed that the cells adjacent to e_1 and e_2 are in distinct components of the dual graph $D(\mathrm{gh}_f(C))$. Hence $C \setminus \{c\}$ is full, contradicting the minimality of C. We conclude that $\mathrm{gh}_f(C)$ has a reflex interior angle at c.

Next we check the remaining condition in Definition 5. If v_1v_2 or v_2v_3 is not a bridge, then this condition is satisfied. If both v_1v_2 and v_2v_3 are bridges (with possibly $v_1v_2 = v_2v_3$), then by our assumption the corresponding dual edges are in distinct components of $D(\mathrm{gh}_f(C))$. In both cases, we have $c \in A_f(C)$. □

Lemma 4 *Let G be a planar straight line multigraph with $b \geq 1$ bridges and $n \geq 3$ vertices in general position in the plane such that all bridges are adjacent to a face f. Let C be a minimal full set of corners such that $\mathrm{gh}_f(C)$ visits every corner at most once. Let x be the number of components of $D(\mathrm{gh}_f(C))$ that contain exactly two nodes. Then G has an embedding preserving augmentation to a 2-edge connected* PSLG *such that*

(i) *we use at most $\lfloor (x + 2b)/3 \rfloor$ new edges, all lying in f;*

(ii) *there is a set $X \subseteq A_f(C)$ of x corners such that for every corner $c \in X$, there is a new edge between c and another block of G.*

Proof. First assume that $\mathrm{gh}_f(C)$ has only two vertices, and so $D(\mathrm{gh}_f(C))$ has exactly 2 nodes, and $x = 1$. Then f cannot be unbounded, since then $\mathrm{gh}_f(C)$ would visit all vertices along $\mathrm{ch}(G)$, and $\mathrm{ch}(G)$ would have at least three vertices. So we may assume that f is bounded. Augment G with a single edge connecting the two vertices of $\mathrm{gh}_f(C)$. This edge connects the two terminal blocks of G (one of which is necessarily the outer block), hence the resulting planar straight line multigraph is 2-edge connected. The number of new edges is one, and $\lfloor (2b + 1)/3 \rfloor \geq 1$ since $b \geq 1$.

Next assume that $\mathrm{gh}_f(C)$ has at least three vertices, i.e., it is a simple polygon. We add new edges for each component of the dual graph $D(\mathrm{gh}_f(C))$, independently. Denote the components of $D(\mathrm{gh}_f(C))$ by D_i, and let b_i be the number of edges in D_i (Fig. 7). Since every bridge of G is adjacent to two cells in a component of $D(\mathrm{gh}_f(C))$, we have $\sum_i b_i \leq b$.

- If D_i is a tree and $b_i \geq 2$, then it has $b_i + 1$ nodes. D_i has a 2-coloring, and a smaller color class contains at most $\lfloor (b_i + 1)/2 \rfloor \leq \lfloor 2b_i/3 \rfloor$ nodes. Complete each cell in a smallest color class to a circuit by an edge of $\mathrm{gh}_f(C)$.

- If D_i is not a tree and $b_i \geq 3$, then it has at most b_i nodes. D_i has a 3-coloring and the two smallest color classes together contain at most $\lfloor 2b_i/3 \rfloor$ nodes. Complete each cell in two smallest color classes to a circuit by an edge of $\mathrm{gh}_f(C)$.

- If $b_i = 1$, then D_i has exactly two nodes. Let e_i denote the counterclockwise first edge of $\mathrm{gh}_f(C)$ along a cell corresponding to D_i. Let v_i be the counterclockwise first vertex of e_i. Complete one of the two corresponding cells of D_i to a circuit with edge e_i. Charge the new edge to $1 = \frac{2}{3}b_i + \frac{1}{3}$, where $\frac{1}{3}$ corresponds to vertex v_i.

Altogether, we have augmented G with at most $\lfloor (x + 2b)/3 \rceil$ new edges. This proves part (i). For part (ii), notice that in the third case the vertices v_i are distinct, and by Proposition 6 they are the apices of distinct corners in $A_f(C)$. Let X be the set of corners c adjacent to edge e_i at vertex v_i for all component D_i with two nodes. Each corner in X is connected to another block of G by a new edge e_i. □

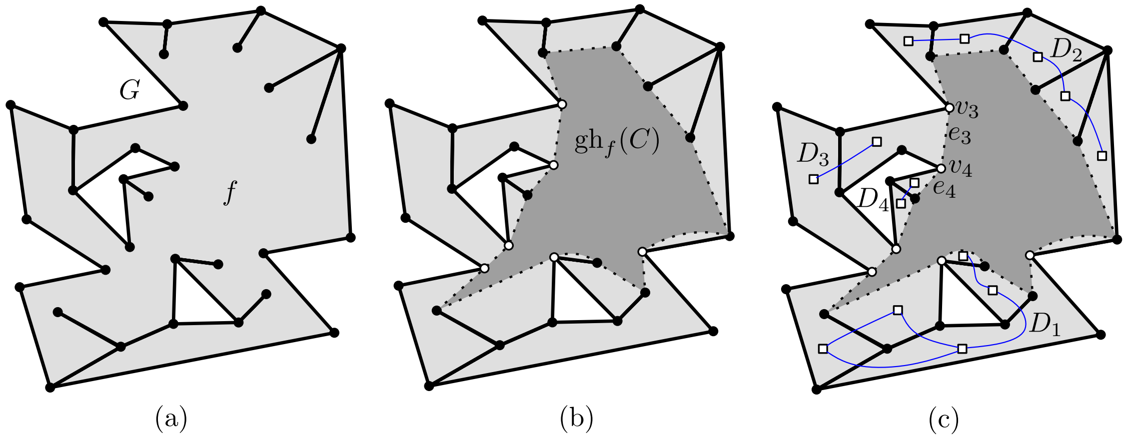


Figure 7: (a) A PSLG with all 11 bridges adjacent to a face f. (b) A geodesic hull $\mathrm{gh}_f(C)$ is a simple polygon for a minimal full set of corners C, the corners in $A_f(C)$ are marked with empty dots. (c) The dual graph $D(\mathrm{gh}_f(C))$ has four components, D_i, $i = 1, 2, 3, 4$.

A key lemma. The following lemma is the key for the proof of Theorem 1. It extends Lemma 4 to the general case where C is not necessarily minimal and $\mathrm{gh}_f(C)$ is not necessarily a simple polygon.

Lemma 5 *Let G be a planar straight line multigraph with $b \geq 1$ bridges and $n \geq 3$ vertices in general position in the plane such that all bridges are adjacent to a face $f \in F(G)$. Let C be a full set of corners, and let $a = |A_f(C)|$. Then G has an embedding preserving augmentation to a 2-edge connected* PSLG *with at most $\lfloor (a + 2b)/3 \rfloor$ new edges, all lying in f.*

Proof. Proceed by induction on $b + |C|$. If $b = 0$, then G is already 2-edge connected, and no new edge is necessary. Assume that $b \geq 1$. Then G has at least two terminal blocks, and so $|C| \geq 2$. It is enough to augment G to a 2-edge connected planar straight line *multigraph* with the specified number of new edges, and then Lemma 1 completes the proof.

We distinguish three cases.

Case 1: C **is not minimal.** In this case, there is a set of corners $C' \subsetneq C$ such that C' is full. Then $|A_f(C')| \leq |A_f(C)|$ and induction completes the proof.

Case 2: C **is minimal and** $\mathrm{gh}_f(C)$ **visits every corner at most once.** In this case, Lemma 4 completes the proof, noting that $x \leq a$ by Proposition 6.

Case 3: C **is minimal and** $\mathrm{gh}_f(C)$ **visits some corner twice.** Let $\mathrm{gh}_f(C) = (c_1, c_2 \ldots, c_k)$, with some repetitions. Suppose w.l.o.g. that $c_1 = c_\ell$, and $(c_1, c_2, \ldots, c_\ell)$ is a maximal subsequence without repetitions, and $c_1 = c_\ell$. Let $C_1 = \{c_1, c_2, \ldots, c_\ell\}$ and $C_2 = \{c_\ell, c_{\ell+1}, \ldots, c_k\}$, where $c_1 = c_{\ell+1} \in C_1 \cap C_2$. We may also assume that (1) if f is unbounded, then C_2 contains all corners on the convex hull, (2) if f is bounded, then C_2 contains the stem corner in C.

The weakly simple polygon $\mathrm{gh}_f(C)$ decomposes into $P_1 = \mathrm{gh}_f(C_1)$, which is a either a 2-gon (a pair of parallel edges) or a simple polygon, and a weakly simple polygon $P_2 = \mathrm{gh}_f(C_2)$. Note that P_1 visits every corner at most once, and P_2 visits corner $c_1 = c_{\ell+1}$ only once.

Since $\mathrm{gh}_f(C)$ visits c_1 twice, c_1 is a reflex corner of G and $\mathrm{gh}_f(C)$ has a reflex interior angle at c_1 by Proposition 5(ii). Let $c_1 = (v_0, v_1, v_2)$, where v_1 is the apex of c_1. By Proposition 5(ii), all edges of $\mathrm{gh}_f(C)$ and G incident to v_1 lie in a halfplane whose boundary contains v_1. Every ray in the complementer halfplane hits some edges or vertices of G (otherwise f would be the unbounded face and $\mathrm{gh}_f(C)$ would not visit c_1 twice). There is a vertex $w \in V(G)$ such that v_1w lies in this complementer halfplane and segment v_1w is compatible with G. The line segment v_1w decomposes face f into two faces, which we denote by f_1 and f_2 (Fig. 8). Assume that $\mathrm{gh}_f(C_1)$ lies in face $\mathrm{cl}(f_1)$, and $\mathrm{gh}_f(C_2)$ lies in $\mathrm{cl}(f_2)$. Note that $\mathrm{gh}_f(C)$ has a reflex interior angle at c_1, but both $\mathrm{gh}_f(C_1)$ and $\mathrm{gh}_f(C_2)$ have a convex interior angle at c_1.

Let G_1 be the PSLG composed of the subgraph of G on the boundary of f_1, and of edge v_1w. Let B_1 be the set of bridges whose relative interior lie in the interior of $\mathrm{cl}(f_1)$, this is the set of bridges of G_1. Let B_0 be the set of bridges of G lying on the boundary of $\mathrm{cl}(f_1)$, and let $B_2 = B \setminus (B_0 \cup B_1)$. Let $b_0 = |B_0|$, $b_1 = |B_1|$, and $b_2 = |B_2|$ with $b = b_0 + b_1 + b_2$. Note that $\mathrm{gh}_f(C_1) = \mathrm{gh}_{f_1}(C_1)$. Furthermore, $\mathrm{gh}_f(C_1)$ visits every terminal block of G_1 and c_1 is the only corner where $\mathrm{gh}_f(C_1)$ has a convex interior angle at a corner of the outer block of G_1. Hence, C_1 is a minimal full set of corners for the PSLG G_1, with a stem corner at c_1.

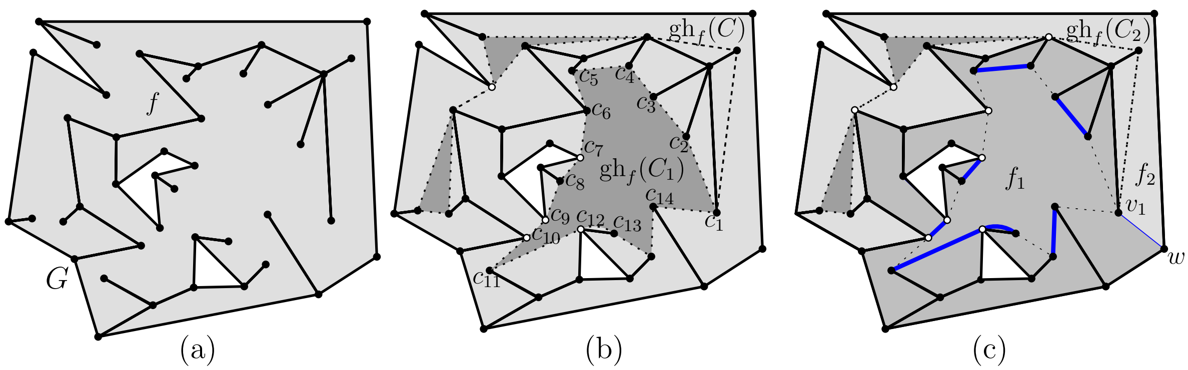


Figure 8: (a) A PSLG with all 26 bridges adjacent to a face f. (b) A geodesic hull $\mathrm{gh}_f(C)$ is a weakly simple polygon for a minimal full set of corners C, the corners in $R(C)$ are marked with empty dots. (c) Corner c_1 splits $\mathrm{gh}_f(C)$ into a simple polygon $\mathrm{gh}_f(C_1)$ and a weakly simple polygon $\mathrm{gh}_f(C_2)$.

Construct an embedding preserving augmentation of G_1 to a 2-edge connected PSLG by Lemma 4, using the minimal full set of corners C_1. If $D(\mathrm{gh}_{f_1}(C_1))$ has x components with exactly two nodes, then we use at most $\lfloor (x + 2b_1)/3 \rfloor$ new edges, and there is a set $X \subseteq A_{f_1}(C_1)$ of x corners connected to another block of G_1 by a new edge.

The new edges augment G to a planar straight line multigraph G' (without the auxiliary edge v_1w). The new edges lie in face f_1 and partition f into several faces. Let f', $f_2 \subset f' \subset f$, denote the face of G' containing $\mathrm{gh}_f(C_2)$. Every bridge of G' is adjacent to f', since the edges in B_1 are no longer bridges in G'. Hence, $\mathrm{gh}_{f'}(C_2) = \mathrm{gh}_f(C_2)$. Note also that f' is bounded if and only if f is bounded. If f' is bounded, then $\mathrm{gh}_{f'}(C_2)$ has a convex interior angle at the stem corner of C (by the choice of P_1). Therefore C_2 is a full set of corners for G'. Denote by b' the number of bridges in G', and let $a' = |A'(C_2)|$.

By induction, G' has an embedding preserving augmentation to a 2-edge connected PSLG with at most $\lfloor (a'+2b')/3 \rfloor$ new edges. To complete the induction step, we need to show

$$\left\lfloor \frac{x+2b_1}{3} \right\rfloor + \left\lfloor \frac{a'+2b'}{3} \right\rfloor \leq \left\lfloor \frac{a+2b}{3} \right\rfloor. \tag{1}$$

It is enough to prove $(x+2b_1)+(a'+2b') \leq a+2b$, or equivalently

$$x+2b_1 \leq (a-a')+2(b-b'). \tag{2}$$

Consider the bridges in B_0 lying on the boundary of $\mathrm{cl}(f_1)$. Let $B'_0 \subset B_0$ be the subset of bridges that remain bridges in G', and let $B^*_0 = B_0 \setminus B'_0$. Denote their cardinalities by $b'_0 = |B'_0|$ and $b^*_0 = |B^*_0|$. Then the bridges of G' are $B' = B'_0 \cup B_2$, with $b' = b'_0 + b_2$. Therefore, $b - b' = b_1 + b^*_0$, and (2) is equivalent to

$$x \leq (a-a') + 2b^*_0. \tag{3}$$

First of all, consider corner c_1. It is a stem corner in C_1 with respect to G_1. It is not a stem corner in C_2, but $\mathrm{gh}_f(C_2) = \mathrm{gh}_{f'}(C_2)$ has a convex corner at c_1. Hence the corner at c_1 is in $A_{f_1}(C_1)$, but not in $A_{f'}(C_2)$. By definition, it is clear that

$$X \subseteq A_{f_1}(C_1) \subseteq A_f(C) \cap C_1 \qquad \text{and} \qquad A_{f'}(C_2) \subseteq (A_f(C) \setminus \{c_1\}) \cap C_2.$$

Ideally, we have $A_{f_1}(C_1) = A_f(C) \cap C_1$ and $A_{f'}(C_2) = (A_f(C) \setminus \{c_1\}) \cap C_2$. In this case, $|A_{f_1}(C_1)| = a - a'$, $X \subseteq A_{f_1}(C_1)$ and so $x \leq a_1$. Therefore $x \leq a - a'$ and (3) follows.

Next we examine the case that $X \subsetneq A_f(C) \cap C_1$ or $A_{f'}(C_2) \subsetneq (A_f(C) \setminus \{c_1\}) \cap C_2$. Consider a corner $c = (u_1, u_2, u_3)$ in $X \subseteq A_{f_1}(C_1)$ but not in $A_f(C) \cap C_1$. Then by Definition 5, both u_1u_2 and u_2u_3 are bridges in G (possibly $u_1u_2 = u_2u_3$), but at most one of them is a bridge is G_1. That is, at least one of u_1u_2 and u_2u_3 is in B_0. However, after adding a new edge connecting $c \in X$ to another block, at most one of u_1u_2 and u_2u_3 is a bridge in G'. That is, u_2 is one endpoint of an edge in B^*_0. Now consider a corner $c = (u_1, u_2, u_3)$ in $A_{f'}(C_2)$ but not in $A_f(C) \cap C_2$. Then both u_1u_2 and u_2u_3 are bridges in B_0, but at most one of them is a bridge in G'. That is, u_2 is again an endpoint of an edge in B^*_0. Every edge in B^*_0 is responsible for changing the status of at most two reflex corners, at most one at each endpoint. Hence (3) follows in general. This completes the induction step in case 3. □

Corollary 3 *Let G be a connected* PSLG *with $n \geq 3$ vertices in general position in the plane such that all bridges are adjacent to a face f. Let b denote the number of bridges of G. Let r denote the number of reflex corners of f adjacent to some non-singleton block of G.*

- *If f is an unbounded face, then G can be augmented to a 2-edge connected* PSLG *by adding at most $\lfloor (2b+r)/3 \rfloor$ new edges, all lying in f.*
- *If f is a bounded face, then G can be augmented to a 2-edge connected* PSLG *by adding at most $\lfloor (2b+r+1)/3 \rfloor$ new edges, all lying in f.*

Proof. Note that $a \leq r$ if f is unbounded, and $a \leq r+1$ if f is bounded. The claims immediately follow from Lemma 5. □

7 Proofs of Theorems 1 and 2

We start with proving Theorem 2, the proof of Theorem 1 is similar but more involved.

Proof of Theorem 2. Let G be a connected PTG with $n \geq 7$ vertices. If G has only one face, then G is a tree, and it has an embedding preserving augmentation with $\lfloor n/2 \rfloor$ new edges by Theorem 3(i). Note that $\lfloor n/2 \rfloor \leq \lfloor (2n-4)/3 \rfloor$ for $n \geq 7$.

Assume now that G has at least two faces. For a face $f \in F(G)$, denote by G_f the PTG formed by the edges and vertices of G along the boundary of f, and let b_f denote the number of bridges of G_f. By Corollary 2, G_f has an embedding preserving augmentation to a 2-edge connected PTG by adding at most $\lceil b_f/2 \rceil$ new edges, all lying in face f. The embedding preserving augmentations of G_f, for all $f \in F(G)$, give an embedding preserving augmentation of G, which is a 2-edge connected PTG since it is the union of 2-edge connected PTGs. Therefore, G has an embedding preserving augmentation to a 2-edge connected PTG with at most $\sum_{f \in F(G)} \lceil b_f/2 \rceil$ new edges.

Next, we transform G to a PTG G' such that $\sum_{f \in F(G)} \lceil b_f/2 \rceil = \sum_{f \in F(G')} \lceil b_f/2 \rceil$, and then we show that $\sum_{f \in F(G')} \lceil b_f/2 \rceil \leq \lfloor (2n-5)/3 \rfloor$. The PTG G' will be a triangulation with some leaves added in some of the triangular faces. We transform the graph G_f for each face $f \in F(G)$ independently. Since G is connected and has at least two faces, at least three edges of G_f are part of a common block. Contract each of the b_f bridges of G_f (with a continuous deformation of the planar embedding as in [23]). Insert $\lfloor b_f/2 \rfloor$ new vertices in the interior of the resulting face, and triangulate it into at least $2\lfloor b_f/2 \rfloor + 1$ triangular faces. Then insert a leaf into $\lceil b_f/2 \rceil$ of these triangular faces. The transformation does not change the number of vertices, and it also does not change the sum $\sum_{f \in F(G)} \lceil b_f/2 \rceil = \sum_{f \in F(G')} \lceil b_f/2 \rceil$.

After the transformation, the resulting graph G' is a triangulation with a leaf added in some distinct triangular faces. If b' denotes the number of bridges (and leaves) in G', then $\sum_{f \in F(G')} \lceil b_f/2 \rceil = \sum_{f \in F(G')} b_f = b'$. A triangulation formed by $n - b' \geq 3$ vertices has exactly $2(n-b') - 4$ faces. Hence $b' \leq 2(n-b') - 4$, and so $b' \leq \lfloor (2n-4)/3 \rfloor$. If G has at least two faces, then $\sum_{f \in F(G)} \lceil b_f/2 \rceil = b' \leq \lfloor (2n-4)/3 \rfloor$. □

Proof of Theorem 1. Let G be a connected PSLG with $n \geq 3$ vertices in general position in the plane. We compute an embedding preserving augmentation of G to a 2-edge connected PSLG by augmenting the subgraph of G on the boundary of every face $f \in F(G)$ to a 2-edge connected PSLG with new edges lying in f. The resulting graph is the union of 2-edge connected PSLGs (one for each face of G), and hence it is a 2-edge connected PSLG. It remains to estimate the total number of new edges.

For a face $f \in F(G)$, denote by G_f the PSLG formed by the edges and vertices of G along the boundary of f, and let b_f denote the number of bridges of G_f. Denote by r_f the number of reflex corners of nonsingleton blocks of G adjacent to f. We combine the upper bounds for the sufficient number of new edges given by Corollaries 1 and 3. For a bounded face $f \in F(G)$, let

$$\kappa(f) = \min\left(b_f, \left\lfloor \frac{2b_f + r_f + 1}{3} \right\rfloor\right).$$

For the unbounded face $f_0 \in F(G)$, let $\kappa(f_0) = \min(b_f, \lfloor (2b_f + r_f)/3 \rfloor)$. By Corollaries 1 and Corollary 3, G_f has an embedding preserving augmentation to a 2-edge connected PSLG with at most $\kappa(f)$ new edges lying in f. Therefore, G has an embedding preserving augmentation to a 2-edge connected PSLG with at most $\sum_{f \in F(G)} \kappa(f)$ new edges.

In the remainder of the proof, we transform G to a PTG G' such that $\sum_{f \in F(G)} \kappa(f) = \sum_{f \in F(G')} \kappa(f)$, and then we show that $\sum_{f \in F(G')} \kappa(f) \leq \lfloor (2n-2)/3 \rfloor$. The PTG G' will be a triangulation with some leaves added in some of the faces. Since G' is not necessarily a PSLG, and reflex corners are not defined in PTGs, we need to extend the definition of $\kappa(f)$ to faces $f \in F(G')$ of the PTG G'. We use the convention (similar to [11]) that all three corners of a bounded triangular face are *convex*, all three corners of an unbounded triangular face are reflex, and the corner at every leaf vertex is reflex. With this convention, let r_f be the number of reflex corners in a face $f \in F(G')$, and the definition of $\kappa(f)$ extends to the faces of G'.

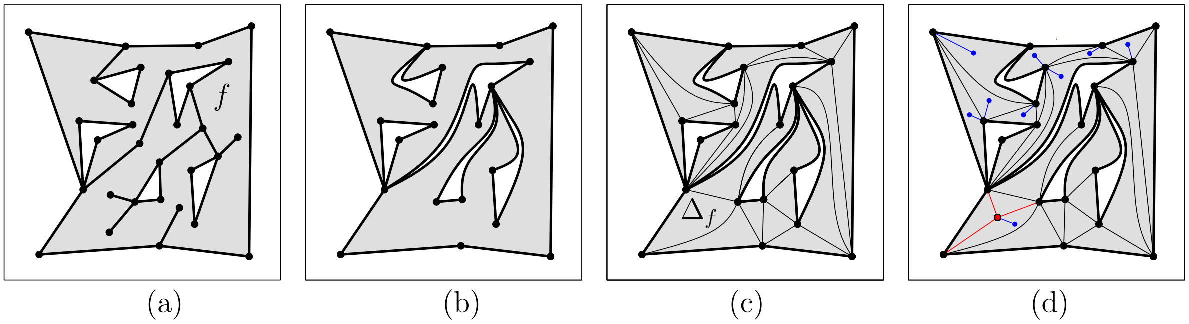


Figure 9: (a) A PSLG G_f on the boundary of a bounded face f with $b_f = 10$ bridges, $r_f = 13$ reflex corners adjacent to non-singleton blocks. (b) Contracting all bridges. (c) Triangulating the resulting face. (d) Adding a leaf in 8 triangles each, adding one Steiner point in the interior of an additional triangle Δ_f, triangulating Δ_f, and adding a leaf in one of the triangles in Δ_f.

Let us first fix a bounded face $f \in F(G)$. (See Fig. 9.) Every vertex of f on the convex hull of f is adjacent to at least two nonbridge edges of f (lying on the outer boundary of $\mathrm{cl}(f)$). At most one reflex corner is adjacent to each vertex, and each reflex corner adjacent to a non-singleton block is incident to at least two nonbridge edges of f. Every nonbridge edge is counted twice (at most once at each endpoints), hence f is adjacent to at least $r_f + 3$ nonbridge edges.

Contract each of the b_f bridges adjacent to f. The resulting face is adjacent to at least $r_f + 3$ edges. Triangulate the resulting face. We obtain at least $r_f + 1$ triangles. If $b_f \leq r_f + 1$, then add a leaf in b_f triangles each. In this case, $\kappa(f) = b_f$; and in the resulting triangulation, b_f triangles each contain a bridge. If $b_f > r_f + 1$, then add a leaf in r_f triangles each; add $\lceil (b_f - r_f - 1)/3 \rceil \geq 1$ Steiner points in the interior of an additional triangle $\Delta_f \subset f$; triangulate Δ_f using the Steiner points into $1 + 2\lceil (b_f - r_f - 1)/3 \rceil$ triangles; and add a leaf in $b_f - r_f - \lceil (b_f - r_f - 1)/3 \rceil = \lfloor (2b_f - 2r_f + 1)/3 \rfloor$ triangles each. In this case, $\kappa(f) = \lfloor (2b_f + r_f + 1)/3 \rfloor$; and in the resulting triangulation, we have $\kappa(f') = 1$ for $r_f + \lfloor 2(b_f - r_f + 1)/3 \rfloor = \lfloor (2b_f + r_f + 1)/3 \rfloor$ triangles. In this transformation, neither the number of vertices nor $\sum_{f \in F(G)} \kappa(f)$ changes.

Now consider the unbounded face. (See Fig. 10.) Every vertex of f on the convex hull of f is adjacent to a reflex corner of f. Each reflex corner adjacent to a non-singleton block is incident to at least two nonbridge edges of f. Hence f is adjacent to at least $r_f \geq 3$ nonbridge edges. Contract each of the b_f bridges adjacent to f. The resulting face is adjacent to at least r_f edges. Triangulate the resulting face: all but one triangles are bounded. The number of bounded triangular faces is at least $r_f - 3$. If $b_f \leq r_f$, then add three leaves in the unbounded face and add a leaf in exactly $b_f - 3$ bounded faces each. In this case, $\kappa(f) = b_f$; and in the resulting triangulation, we have $\kappa(f_0') = 3$ for the unbounded face f_0', and $\kappa(f') = 1$ for $b_f - 3$ bounded triangles. If $b_f > r_f$, then add a leaf in exactly $r_f - 3$ bounded faces each; add $\lceil (b_f - r_f)/3 \rceil$ Steiner points in the unbounded face f_0; triangulate f_0 using the Steiner points into an unbounded triangle and $2\lceil (b_f - r_f)/3 \rceil$ bounded triangles; add a leaf in $b_f - r_f - \lceil (b_f - r_f)/3 \rceil = \lfloor 2(b_f - r_f)/3 \rfloor$ bounded triangles within f_0 each; and add three leaves in the unbounded triangle in f_0. In this case, $\kappa(f) = \lfloor (2b_f + r_f)/3 \rfloor$; and in the resulting triangulation, we have $\kappa(f_0') = 3$ for the unbounded face, and $\kappa(f') = 1$ for $r_f - 3 + \lfloor 2(b_f - r_f)/3 \rfloor = \lceil (2b_f + r_f)/3 \rceil - 3$ bounded triangles. In this transformation, neither the number of vertices nor $\sum_{f\in F(G)}\kappa(f)$ changes.

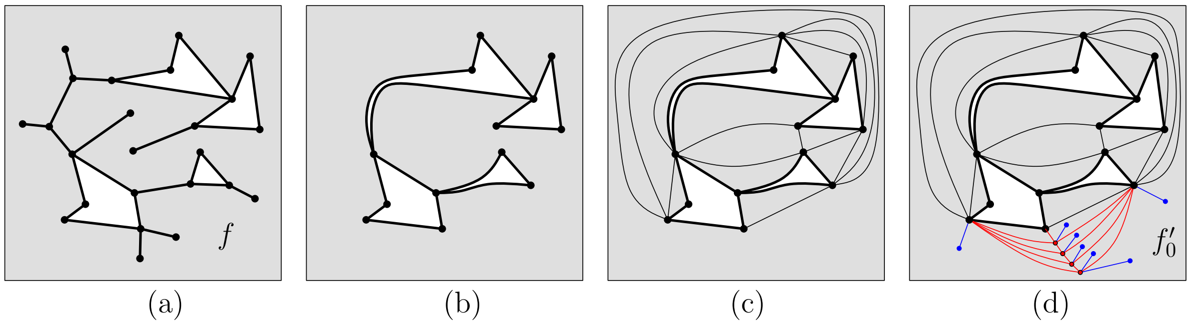


Figure 10: (a) A PSLG G_f on the boundary of an unbounded face f with $b_f = 11$ bridges, $r_f = 5$ reflex corners adjacent to non-singleton blocks. (b) Contracting all bridges. (c) Triangulating the resulting face. (d) Adding 4 Steiner points in the interior of the unbounded face f'_0, triangulating f'_0, and adding a leaf in 4 bounded triangles within f_0 each, and adding three leaves in the unbounded face.

After the transformation, the resulting PTG G' is a triangulation where the outer face has exactly 3 reflex corners adjacent to a non-singleton block and contains 3 bridges; every bounded face is either a triangle or a triangle containing a leaf. For the outer face, we have $(b_{f'_0}, r_{f'_0}) = (3,3)$ and so $\kappa(f'_0) = 3$. For every bounded face f, we have $(b_f, r_f) = (0,0)$ and $\kappa(f) = 0$ if f is a triangle, whereas $(b_f, r_f) = (1,0)$ and $\kappa(f) = 1$ if f is a triangle containing a leaf. Therefore, $\sum_{f\in F(G)}\kappa(f) = \sum_{f\in F(G')}\kappa(f) = b'$, the number of bridges of G'. Recall that 3 leaves lie in the unbounded face and $b'-3$ leaves lie in bounded faces. A triangulation formed by $n-b'$ vertices has exactly $2(n-b')-5$ bounded faces. Hence $b'-3 \le 2(n-b')-5$, and so $b' \le \lfloor (2n-2)/3 \rfloor$. We conclude that $\sum_{f\in F(G)}\kappa(f) = b' \le \lfloor (2n-2)/3 \rfloor$, as required. □

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