Graphs that Admit Right Angle Crossing Drawings

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Abstract

We consider *right angle crossing (RAC) drawings* of graphs in which the edges are represented by polygonal arcs and any two edges can cross only at a right angle. We show that if a graph with n vertices admits a RAC drawing with at most 1 bend or 2 bends per edge, then the number of edges is at most $6.5n$ and $74.2n$, respectively. This is a strengthening of a recent result of Didimo *et al.*

1 Introduction

The core problem in graph drawing is finding good and easily readable drawings of graphs. Recent cognitive experiments [10, 11] show that poly-line graph drawings with orthogonal crossings and a small number of bends per edge are just as readable as planar drawings. Motivated by these findings, Didimo *et al.* [7] studied the class of graphs which have a polyline drawing where crossing edges meet at a right angle. Such a drawing is called a *right angle crossing* drawing, or *RAC drawing*, for short.

The interior vertices of a polygonal arc are called *bend*s. We say that a planar representation of a graph is an RAC_b *drawing*, for some $b \in \mathbb{N}_0$, if the vertices are drawn as points, the edges are drawn as polygonal arcs with at most b bends joining the corresponding vertices, and any two polygonal arcs cross at a right angle (and not at a bend). Let R_b, $b \in \mathbb{N}_0$, be the class of graphs that admit a RAC_b drawing. It is clear that $R_b \subseteq R_{b+1}$ for all $b \in \mathbb{N}_0$. Didimo *et al.* [7] showed that every graph is in R_3, hence $R_3 = R_b$ for all $b \geq 3$. They proved that every graph with $n \geq 4$ vertices in R_0 has at most $4n - 10$ edges, and this bound is best possible. They also showed that a graph with n vertices in the classes R_1 and R_2 has at most $O(n^{4/3})$ and $O(n^{7/4})$ edges, respectively.

Results. We significantly strengthen the above results, and show that every graph with n vertices in R_1 and R_2 has at most $O(n)$ edges, and that the classes R_0, R_1 and R_2 are pairwise distinct.

Theorem 1. *A graph G with n vertices that admits a RAC_1 drawing has at most $6.5n - 13$ edges.*

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Theorem 2. *A graph G with n vertices that admits a RAC_2 drawing has less than $74.2n$ edges.*

We use two quite different methods to prove our main results. In Section 2, we use the so-called discharging method to prove Theorem 1. In Section 3, we define *block graphs* on the crossing edges, and use the Crossing Lemma to prove Theorem 2. Each method gives a linear bound for the number of edges for graphs in both R_1 and R_2, however, they would each give weaker constant coefficients for the other case (i.e., for R_2 and R_1, respectively).

We complement our upper bounds with lower bound constructions in Section 4. We construct graphs with n vertices in the classes R_1 and R_2 with $4.5n - O(\sqrt{n})$ and $7.8\dot{3}n - O(\sqrt{n})$ edges, respectively. Combined with Theorems 1 and 2, they show that $R_0 \neq R_1$ and $R_1 \neq R_2$.

Related Work. Angelini *et al.* [4] proved that every graph of maximum degree 3 admits a RAC_1 drawing, and every graph of maximum degree 6 admits a RAC_2 drawing. They also show that some planar directed graphs do not admit straight line *upward* RAC drawings.

A natural generalization of RAC drawings with straight line edges is given by Dujmović *et al.* [8]. They define *α-angle crossing* (αAC) drawings to be straight line graph drawings where every pair of crossing edges intersect at an angle at least α. In line with the results by Didimo *et al.* [7] on RAC drawings, they prove upper bounds on the number of edges for αAC graphs and give lower bound constructions. Specifically, they prove that the number of edges in an αAC graph is at most $(\pi/\alpha)(3n-6)$ for $0 < \alpha < \pi/2$ and at most $6n - 12$ for $2\pi/5 < \alpha < \pi/2$. In addition, they give lower bound constructions based on the square and hexagonal lattices for $\alpha = \pi/k$, $k = 2, 3, 4, 6$. Di Giacomo *et al.* [6] also generalize RAC drawings in this way and call the minimum angle of any crossing the *crossing resolution*.

Preliminaries. The *crossing number* of a graph G, denoted cr(G), is the minimum number of edge crossings in a drawing of G in the plane. The Crossing Lemma, due to Ajtai *et al.* [3] and Leighton [12], establishes a lower bound for $\text{cr}(G)$ in terms of the number of vertices and edges. The strongest known version is due to Pach *et al.* [13].

Lemma 1. *[13] Let G be a graph with n vertices and m edges. If $m \geq \frac{103}{6}n \approx 17.167n$, then*

$$\text{cr}(G) \geq c \cdot \frac{m^3}{n^2}, \text{ where } c = \frac{1024}{31827} \approx 0.032. \tag{1}$$

Let G be a graph with n vertices and m edges, and let D be a RAC_b drawing of G. If there is no confusion, we make no distinction between the vertices (edges) of G and the corresponding points (polylines) of D.

A *plane (multi-)graph* is a (multi-)graph drawn in the plane without any edge crossings. The *faces* of a plane (multi-)graph are the connected components of the complement of G. Let D be a drawing of a graph $G = (V, E)$. A *rotation system* at a vertex $v \in V$ in drawing D is the (clockwise) circular order in which the edges leave v. A *wedge* at a vertex v in D is an ordered pair of edges (e, e') incident to v that are consecutive in its rotation system. A face f in D is *adjacent* to a wedge (e, e') if e, v, and e' are consecutive in a counterclockwise traversal of the boundary of f. Every wedge is adjacent to a unique face in D. The *size* of a face is the number of edges (counted with multiplicity) on the boundary of f.

2 RAC Drawings with One Bend per Edge

Discharging. We apply a discharging method reminiscent to that of Ackerman and Tardos [2] to prove Theorem 1. This method was apparently introduced by Wernicke [14], but it gained considerable attention only after it was extensively used in the first valid proof of the famous Four Color Theorem [5]. Since then, it was instrumental in deriving various types of results in structural graph theory, see e.g. [9]. Dujmović *et al.* [8] applied the discharging method for an alternative proof for the upper bound of $4n-10$ on the number of edges in a graph on n vertices that admits a straight line RAC (i.e. RAC_0) drawing, originally due to Didimo *et al.* [7].

Proof of Theorem 1. Let $G=(V,E)$ be a graph in R_1. Fix a RAC_1 drawing D of G that minimizes the number of edge crossings. Partition G into two subgraphs $G_0=(V,E_0)$ and $G_1=(V,E_1)$, where $E_0\subseteq E$ is the subset of crossing free edges and $E_1\subseteq E$ is the subset of edges with at least one crossing. Since G_0 is planar, it has at most $|E_0|\leq 3n-6$ edges.

Let C be the set of crossing points in D. We construct a plane multigraph $G'=(V',E')$ as follows: the vertices $V'=V\cup C$ are the vertices in V and all crossings in C; the edges are polygonal arcs between two consecutive vertices along the edges in E_1. That is, the edges in E' are obtained by subdividing the edges in E_1 at crossing points. Since the bends of edges in E_1 are not vertices in G', they are bends of some edges in E'. Denote by F' the set of faces of G'. A bend of an edge determines two angles: a *convex* and a *reflex* angle. We say that face $f\in F'$ is *adjacent* to a convex (resp. reflex) bend, if it has a convex (resp. reflex) interior angle at a bend point. A bounded face of size two is called a *lens*, and is adjacent to two parallel edges. A bounded face of size 3 is called a *triangle*.

Lemma 2. *Every lens $f\in F'$ is adjacent to a convex bend. If it is adjacent to exactly one convex bend, then it is incident to one vertex in C and V each, and adjacent to one convex bend and one reflex bend.*

Proof. Every lens $f\in F'$ is drawn as a simple polygon whose vertices are the incident vertices in V' and adjacent bends. Every simple polygon has at least 3 convex interior angles. A lens is incident to exactly two vertices in V', so it must have a convex interior angle at an adjacent bend.

Let $f\in F'$ be a lens adjacent to exactly one convex bend. Since every edge in E_1 crosses some other edges, no two adjacent vertices in V' are in V. At each vertex in C, the incident faces have 90° interior angles since D is a RAC drawing. If both vertices of lens f are in C with 90° interior angles, then f must have two convex bends. So, f is incident to one vertex in C and V each. If f has only one bend (see Fig. 1(a)), then we can redraw the edge $e\in E$ containing this bend in D with one fewer crossings (eliminating the crossing incident to f), which contradicts the choice of the RAC_1 drawing D. So f must be adjacent to a reflex bend (see Fig. 1(b)) as well. □

Lemma 3. *Every triangle $f\in F'$, which is not the outerface, is adjacent to a convex bend.*

Proof. A triangle $f\in F'$ has three vertices in $V'=V\cup C$, and each of its three edges is a polygonal arc with 0 or 1 bends. Since every edge in E_1 crosses some other edges, no two adjacent vertices in V' are in V. That is, at least two vertices of f are in C, with an inner angle of 90°. If f is adjacent to $k\in\{0,1,2,3\}$ bends (at most one bend per edge), then f is a simple polygon with $k+3$ vertices, and so the sum of its interior angles is $(k+1)180^\circ$. If all k bends are reflex, then the sum of interior angles would be more than $90^\circ+90^\circ+k\cdot 180^\circ=(k+1)180^\circ$. □

Lemma 4. *We have* $|E_1| \le 4n - 8$.

Proof. Assume without loss of generality that $G' = (V', E')$ is connected. For a face $f \in F'$, let s_f be the size of f. For a vertex $v \in V' = V \cup C$, let d_v denote the degree of v in G'. We put a charge $\mathrm{ch}(v) = d_v - 4$ on each vertex $v \in V'$, and a charge $\mathrm{ch}(f) = s_f - 4$ on each face $f \in F'$. By Euler's formula the sum of all charges is

$$\sum_{v \in V'} \mathrm{ch}(v) + \sum_{f \in F'} \mathrm{ch}(f) = -8. \tag{2}$$

Indeed, $\sum_{v \in V'}(d_v - 4) + \sum_{f \in F'}(s_f - 4) = 2|E'| - 4|V'| + 2|E'| - 4|F'| = -8$.

Since the charge at a vertex $v \in C$ is 0, we have

$$\sum_{v \in V} \mathrm{ch}(v) + \sum_{f \in F'} \mathrm{ch}(f) = -8. \tag{3}$$

In what follows, we redistribute the charges in G' such that the total charge of all vertices and faces remains the same. The redistribution is done in two steps. In step 1, we move charges from some vertices to some faces; and in step 2 we move charges from some faces to some other faces. Our goal is to ensure that all faces have non-negative charges after the second step.

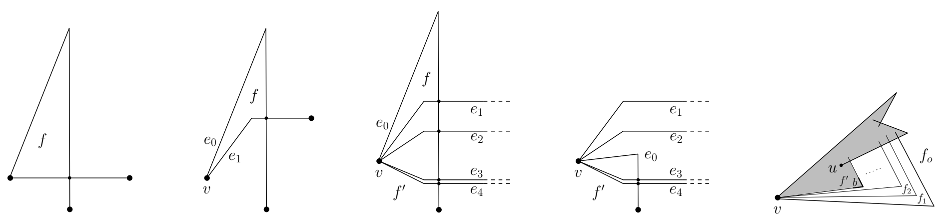

Figure 1: (a) lens f that can be redrawn, (b) lens f having only one convex bend on its boundary, (c) situation when G could be redrawn with fewer crossings, (d) its redrawing (i=4), and (e) triangular outerface

Step 1. For every edge $e \in E_1$ with one bend, we discharge $\frac{1}{2}$ unit from each of the two endpoints of e to the face adjacent to the convex bend of e. The new charge at every vertex $v \in V$ is $\mathrm{ch}'(v) \ge \frac{1}{2}d_v - 4$. Since every face in F' of size at least 4 receives a non-negative charge already at the beginning, it is enough to take care of the triangles and lenses (bounded faces of size 3 and 2), whose initial charge was -1 and -2, respectively.

By Lemma 3, each triangle $f \in F'$ except the outerface is adjacent to a convex bend, and so its charge has increased by at least 1 in step 1. Its new charge $\mathrm{ch}'(f)$ is at least 0. Similarly, if a lens $f \in F'$ is adjacent to two convex bends, then its charge after step 1 is 0. Hence, the only possible faces whose new charge is still negative are the outerface and the lenses adjacent to exactly one convex bend.

Step 2. In order to increase the charge of the outerface and lenses with exactly one convex bend from -1 to 0, we perform the second discharge step. Note that in the first step we have increased the charge of some faces of size 4 or higher (which was unnecessary), so we can now divert the "wasted" charge to faces with negative charge.

Let f be a lens with exactly one convex bend. By Lemma 2, f is incident to one vertex $v \in V$ and one in C, and it is adjacent to one convex bend and one reflex bend. Let $e_0, e_1, \ldots e_{d_v-1}$ (see Fig 1(c)) denote the edges in E_1 incident to v listed according to the rotation system at v (clockwise) such that the wedge (e_0, e_1) is adjacent to face f. We may assume without loss of generality that e_0 has a convex bend and e_1 has a reflex bend adjacent to f. Let $i \in \mathbb{N}$ be the smallest integer such that the wedge (e_i, e_{i+1}) is adjacent to the face, let us denote it by f', of size at least 4. It is easy to see that i is well-defined, since every edge in E_1 participates in a crossing.

We show that f' is adjacent to the convex bend of edge e_i. Any wedge (e_j, e_{j+1}), $1 \le j \le i-1$, must be adjacent to a triangle bounded by parts of the edges e_j, e_{j+1}, and e_0. Since the (convex) bend of e_0 is adjacent to f, all these triangles are adjacent to a straight line portion of e_0. If any of these triangles is adjacent to the convex bend of e_j and a convex bend or no bend of e_{j+1}, then we can redraw edge e_0 to obtain a RAC_1 drawing of G with fewer crossings, eliminating the crossing incident to f (Figs. 1(c) and 1(d)). So the triangle at any wedge (e_j, e_{j+1}), $1 \le j \le i-1$, is adjacent to the reflex bend of e_{j+1}. Hence f is adjacent to the convex bend of e_i.

Move 1 unit of charge (corresponding to the convex bend of e_i) from f' to f. This increases the charge of f to 0. Since the size of f' is at least 4, its charge remains non-negative. It is also clear that the charge corresponding to the convex bend of e_i is diverted to exactly one lens from f'.

It remains to make sure that the outerface f_o gets non-negative charge in the end as well. If f_o has a negative charge after Step 2, then it is triangle. It must have exactly one vertex v from V, otherwise three of its vertices are crossings each contributing $\frac{3}{2}\pi$ to the sum of the inner angles of the polygon which is the complement of the interior of f_o. Thus, f_o would have at least four bends, which is impossible. Moreover, f_o must be adjacent to three reflex bends, i.e. it looks like f_o on Figure 1(e). Then at least one of the inner faces adjacent to bends of f_o on edges incident to v is not a lens. Let f_1 denote such a face. We inductively define f_{i+1} for $i > 1$: If f_i is not a triangle $f_{i+1} = f_i$. If f_i is a triangle we define f_{i+1} as follows. Let f_{i+1} denote the face on the opposite side of the reflex bend of f_i. The definition of f_{i+1} is correct, since the sum of the interior angles in the grey polygon in Figure 1(e) is 4π. Eventually some $f_i = f'$ has at least four vertices and one unit of charge of the bend between f_i and f_{i-1} can be diverted to the outerface. The charge at b has not been moved in Step 2.

After the second step of redistribution, every face in D' has a non-negative charge. Let $\mathrm{ch}''(v)$ and $\mathrm{ch}''(f)$ denote the charge at each vertex $v \in V$ and $f \in F'$ after step 2. We have

$$|E_1| - 4n = \sum_{v \in V} \left(\frac{1}{2} d_v - 4 \right) \le \sum_{v \in V} \mathrm{ch}''(v) \le \sum_{v \in V} \mathrm{ch}''(v) + \underbrace{\sum_{f \in F'} \mathrm{ch}''(f)}_{\ge 0} = -8.$$

By reordering the terms in the above inequality, we have $|E_1| \le 4n - 8$, as required □

At this point we have already proved that the number of edges in G is no more than $|E_0| + |E_1| \le (3n-6) + (4n-8) = 7n - 14$.

We can improve this bound by applying Lemma 4 independently in each face of the plane graph $G_0 = (V, E_0)$, whose edges are the crossing-free edges in E. Notice that each edge in E_1 is fully contained in exactly one face of G_0. Let F_0 be the set of faces of G_0, and let d_f denote the number of vertices of a face $f \in F_0$. By Lemma 4, each face $f \in F_0$ contains at most $4d_f - 8$ edges of E_1, and it obviously contains no edges of E_1 if f is a triangle (i.e., $d_f = 3$). Summing this upper bound over all faces of G_0, we have

$$|E_1| \leq \sum_{f \in F_0, d_f > 3} (4d_f - 8). \tag{4}$$

Lemma 5. *If a plane graph $G_0 = (V, E_0)$ has n vertices and $3n - 6 - k$ edges, then*

$$\sum_{f \in F_0, d_f > 3} (4d_f - 8) \leq 8k. \tag{5}$$

Proof. Denote by $\tau(G_0)$ the sum on the left hand side of (5). We proceed by induction on k. For $k = 0$, the plane graph G_0 is a triangulation and $\tau(G_0) = 0$.

Assuming that the lemma holds for $k \geq 0$, we show that it holds for $k' = k+1$. Let G_0 be a plane graph with n vertices and $3n-6-k'$ edges. G_0' can be obtained by removing an edge e from a plane graph G_0 with $3n - 6 - k$ edges, for which $\tau(G_0) \leq 8k$ by induction. If edge e is a bridge, then we have $\tau(G_0') = \tau(G_0) \leq 8k < 8k'$. Otherwise the removal of e merges two adjacent faces of G_0, say f_1 and f_2. If none of f_1 and f_2 is a triangle, then $4d_f - 8 = 4(d_{f_1} + d_{f_2} - 2) - 8 = (4d_{f_1} - 8) + (4d_{f_2} - 8)$, and so $\tau(G_0') = \tau(G_0) \leq 8k < 8k'$. If f_1 is a triangle and f_2 is a face of size more than three, then $4d_f - 8 = (4(d_{f_2} + 1) - 8) + 4$, and so $\tau(G_0') \leq \tau(G_0) + 4 \leq 8k + 4 < 8k'$. If both f_1 and f_2 are triangles, then $4d_f - 8 = 4 \cdot 4 - 8 = 8$, and $\tau(G_0') \leq \tau(G_0) + 8 \leq 8k + 8 = 8k'$. This completes the induction step, hence the proof of Lemma 5. □

We have two upper bounds for m, the number of edges in G. Lemma 4 gives $m \leq |E_0| + |E_1| \leq (3n-6-k)+(4n-8) = 7n-k-14$, and Lemma 5 gives $m \leq |E_0|+|E_1| \leq (3n-6-k)+8k = 3n+7k-6$. Therefore, we have $m \leq \max_{k \in \mathbb{N}_0} \min(7n - k - 14, 3n + 7k - 6) = 6.5n - 13$, which is attained for $k = n/2 - 1$. This completes the proof of Theorem 1. □

3 RAC Drawings with Two Bends per Edge

Block graphs. The main tool in the proof of Theorem 2 is the block graph of a RAC drawing and the Crossing Lemma. Note that the block graph we define here is different than block graphs defined in the context of 2-connectivity. Let D be a RAC_2 drawing of a graph $G = (V, E)$. Every edge is a polygonal arc that consists of line segments. Without loss of generality, we assume that every edge has two bends so that each edge has two *end segments* and one *middle segment*. A *block* of D is a connected component in the union of pairwise parallel or orthogonal segments in D. Formally, we define a binary relation on the segments in the polygonal arcs in the drawing D: two segments are related if and only if they cross. The transitive closure of this relation is an equivalence relation. We define a block of D as the union of all segments in an equivalence class. Since the union of crossing edges is connected, every block is a connected set in the plane. Furthermore, all segments in a block have at most two different (and orthogonal) orientations.

By Lemma 1, if $m \geq \frac{103}{6}n$, then the average number of crossings per segment is at least

$$\frac{2c}{3} \cdot \frac{m^2}{n^2},$$

where $c = 1024/31827 \approx 0.032$. We say a segment is *heavy* if it crosses at least $\beta c \frac{m^2}{n^2}$ other segments, where $0 < \beta < 2/3$ is the *heaviness parameter* specified later. A block is *heavy* if it contains a heavy segment.

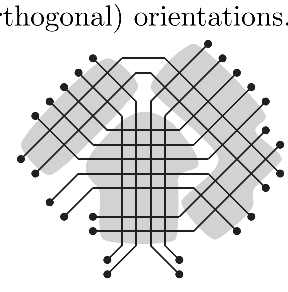

Figure 2: A RAC_2 drawing of a graph and its heavy blocks

We define the *block graph* $B(D)$ as a bipartite multi-graph whose two vertex classes are the vertices in V and the heavy blocks in D.
The block graph has an edge between a vertex $v \in V$ and a heavy block for every segment incident to v and contained in the heavy block (Fig. 2). Note that if a heavy block consists entirely of middle segments, it is not adjacent to any vertex in $B(D)$.

Lemma 6. *If D is a RAC drawing of a graph, then the block graph $B(D)$ is planar.*

Proof. Recall that a heavy block u is a connected set which is incident to all vertices of G that are adjacent to u in $B(D)$. For every heavy block u, replace each crossing with a dummy vertex, and let $T_u \subseteq u$ be a spanning tree of the vertices of G incident to heavy block u. We can construct a planar embedding of $B(D)$. The vertices of G are represented by the same point as in D. Each heavy block u is represented by an arbitrary point r_u in the relative interior of T_u. If vertex v of G is adjacent to a heavy block u, then connect v and r_u by a Jordan arc that closely follows the shortest path between v and r_u in the tree $T_u \subseteq u$. Since shortest paths in a tree do not cross, we can draw the edges successively without crossings. □

Denote by H the number of heavy blocks in D. The block graph $B(D)$ is bipartite and planar, with $H + n$ vertices. If it is *simple*, then it has at most $2(H + n) - 4$ edges. However, $B(D)$ is not necessarily simple: up to four segments of a heavy block may be incident to a vertex v in D.

Lemma 7. *The block graph $B(D)$ has less than $2H + 5n$ edges.*

Proof. Assume that two segments in a heavy block are incident to the same vertex v. Since the block is connected, there is a closed curve γ passing through v and the two segments such that all other blocks lie either in the interior or in the exterior of γ. Hence, multiple edges cannot interleave in the rotation order of a vertex v. Note also that segments in a block are pairwise parallel or orthogonal. It follows that $B(D)$ becomes a *simple* bipartite plane graph after removing at most 3 duplicate edges at each vertex of D. That is, after removing up to $3n$ edges, the remaining simple bipartite plane graph has at most $2(H + n) - 4$ edges. □

Let S denote the number of segments that participate in some heavy block of D. Every heavy block contains at least one heavy segment and all other segments it crosses. That is, a heavy block contains more than $\beta cm^2/n^2$ segments. Since every segment belongs to a unique block, we have

$$H < \frac{S}{\beta cm^2/n^2} = \frac{Sn^2}{\beta cm^2}. \tag{6}$$

The following lemma reformulates the Crossing Lemma for heavy segments in RAC_2 drawings. We show that if a graph G has sufficiently many edges, then a constant fraction of edges must have a segment in some heavy block in a RAC_2 drawing of G.

Lemma 8. *Let D be a RAC_2 drawing of graph G with $m \geq \frac{103}{6}\sqrt{2/(3\beta)}n$ edges. If one can delete xm edges from D, for some $0 < x < 1$, such that every remaining edge segment crosses less than $\beta cm^2/n^2$ others, then $x > 1 - \sqrt{3\beta/2}$.*

Proof. Suppose xm edges were deleted from D to obtain D', a drawing such that every edge segment crosses less than $\beta cm^2/n^2$ other segments. Let G' be the graph associated with D'. The number of remaining edges is $|E(G')| = m - xm = (1-x)m$. If $(1-x)m \geq \frac{103}{6}n$, then the Crossing Lemma gives $\mathrm{cr}(G') \geq c \cdot \frac{(1-x)^3 m^3}{n^2}$, so the average number of crossings per segment in G' is at least

$$\frac{2\mathrm{cr}(G')}{3(1-x)m} \geq \frac{2c}{3} \cdot \frac{(1-x)^2 m^2}{n^2}.$$

Every segment in D' crosses less than $\beta c m^2/n^2$ others in D'. Comparing the upper and lower bounds for the average number of crossings per segment, we have

$$\frac{2c}{3} \cdot \frac{(1-x)^2 m^2}{n^2} < \beta c \frac{m^2}{n^2} \;\Rightarrow\; (1-x)^2 < 3\beta/2 \;\Rightarrow 1 < x + \sqrt{3\beta/2}.$$

If, however, $(1-x)m < \frac{103}{6}n$ but $m \geq \frac{103}{6}\sqrt{2/(3\beta)}n$, then we have again $x > 1 - \sqrt{3\beta/2}$. □

Lemma 8 immediately gives a lower bound on S, the number of segments participating in heavy blocks.

Lemma 9. *Let D be a RAC$_2$ drawing of graph G. If $m \geq \frac{103}{6}\sqrt{2/(3\beta)}n$, then $S > (1 - \sqrt{3\beta/2})m$.*

Proof. Let E_1 be the set of edges containing a segment that participate in some heavy block in D. Clearly, we have $|E_1| \leq S$. If all edges of E_1 are deleted from D, then every remaining segment crosses less than $\beta c m^2/n^2$ others. By Lemma 8, we have $S \geq |E_1| > (1 - \sqrt{3\beta/2})m$. □

Proof of Theorem 2. We set the heaviness parameter to $\beta = 0.062$. If $m \geq \frac{103}{6}\sqrt{2/(3\beta)}n > 56n$, then we can use Lemmas 8 and 9, otherwise $m \leq 56n$ and our proof is complete. Let D be a RAC$_2$ drawing of G. Recall that every edge has two *end segments* and one *middle segment*. Let αS be the number of end segments that participate in heavy blocks, where $0 < \alpha < 1$. The number of middle segments is m, which is a trivial upper bound on the middle segments that participate in heavy blocks. So the total number of segments in heavy blocks is at most $S \leq m + \alpha S$, which gives $S \leq \frac{1}{1-\alpha}m$.

In each heavy block, the segments can be partitioned into two sets of pairwise parallel segments. If we delete all edges that contain some segment in the smaller set of each heavy block, then the remaining segments are not heavy anymore. That is, by deleting at most $\frac{S}{2} \leq \frac{1}{2(1-\alpha)}m$ edges, we obtain a RAC$_2$ drawing with no heavy edge segment. By Lemma 8, we have

$$\frac{1}{2(1-\alpha)} > 1 - \sqrt{\frac{3\beta}{2}} \;\Rightarrow\; \frac{1}{1 - \sqrt{3\beta/2}} > 2(1-\alpha),$$

which implies

$$\alpha > 1 - \frac{1}{2(1 - \sqrt{3\beta/2})} \tag{7}$$

The block graph $B(D)$ has αS edges, since an edge in $B(D)$ exists if and only if a vertex of G is incident to an end segment in a heavy block. From Lemma 7, we have an upper bound on the number of edges in $B(D)$, which gives $\alpha S < 2H + 5n$. Using $S > (1 - \sqrt{3\beta/2})m$ from Lemma 9, the upper bound on H from (6), and the lower bound on α from (7), we obtain

$$\begin{aligned}\alpha S < 2H + 5n \;\Rightarrow\; & \alpha S < \frac{2S}{\beta c}\cdot\frac{n^2}{m^2} + 5n\\ \Rightarrow\; & \left(\alpha - \frac{2}{\beta c}\cdot\frac{n^2}{m^2}\right)\left(1-\sqrt{\frac{3\beta}{2}}\right)m < 5n\\ \Rightarrow\; & 0 < \frac{2-2\sqrt{3\beta/2}}{\beta c}\cdot\left(\frac{n}{m}\right)^2 + 5\cdot\left(\frac{n}{m}\right) - \alpha\left(1-\sqrt{\frac{3\beta}{2}}\right)\\ \Rightarrow\; & 0 < \frac{2-\sqrt{6\beta}}{\beta c}\cdot\left(\frac{n}{m}\right)^2 + 5\cdot\left(\frac{n}{m}\right) - \left(\frac{1}{2}-\sqrt{\frac{3\beta}{2}}\right).\end{aligned}$$

This is a quadratic inequality in n/m. Since $\sqrt{3\beta/2} < 1/2$, the constant term is negative, and the two roots have opposite signs. Therefore, we have

$$\frac{n}{m} > \frac{\beta c}{2(2-\sqrt{6\beta})}\left(-5+\sqrt{25+\frac{4}{\beta c}\left(2-\sqrt{6\beta}\right)\left(\frac{1}{2}-\sqrt{\frac{3\beta}{2}}\right)}\right).$$

This is maximized for $\beta = 0.062$, and gives $m < 74.2n$. □

4 Lower Bound Constructions

We complement the upper bounds in Theorems 1 and 2 with lower bound constructions. We construct an infinite family of graphs which admit RAC_1 drawings and $4.5n - O(\sqrt{n})$ edges. This shows that $R_0 \neq R_1$ since every graph in R_0 has at most $4n-10$ edges [7]. Let the vertices of G be points of the hexagonal lattice clipped in a square (Fig. 3). The edges of G are the hexagon edges and 6 diagonals with a bend in each hexagon. The diagonals connect every other vertex in the hexagon, and make a 75° angle with the side of the hexagon, and so they cross in right angles. The vertex degree is $3 + 3\cdot 2 = 9$ for all but at most $O(\sqrt{n})$ lattice points around the bounding box. Hence the number of edges is $4.5n - O(\sqrt{n})$.

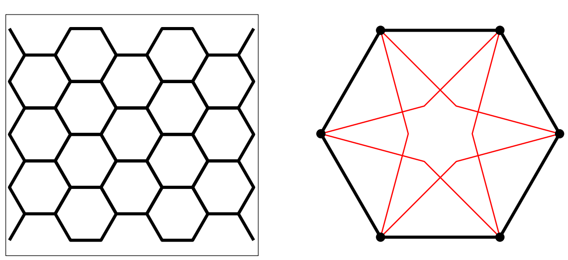

Figure 3: Lower bound construction for a RAC_1 drawing in a hexagonal lattice

We also construct an infinite family of graphs which admit RAC_2 drawings and $7.8\dot{3}n - O(\sqrt{n})$ edges. This shows that $R_1 \neq R_2$ since every graph in R_1 has at most $6.5n-13$ edges by Theorem 1. Let the vertices of G be the vertices of an Archimedean tiling (12,12,3) clipped in a square. Refer to Fig. 4. In the tiling (12,12,3), we can assign two triangles to each 12-gon. The edges of G are the edges of the tiling, a 6-regular graph of diagonals in each 12-gon, and two edges per 12-gon that go to vertices of the two adjacent triangles. The tiling and the diagonals of the 12-gons generate a vertex degree of $3 + 2 \cdot 6 = 15$ at all but at most $O(\sqrt{n})$ vertices (due to the boundary effect). The additional two edges between adjacent 12-gons and triangles increase the average degree to $15 + \frac{2}{3} - O(1/\sqrt{n})$. Hence the number of edges is $\frac{47}{6}n - O(\sqrt{n}) = 7.8\dot{3}n - O(\sqrt{n})$.

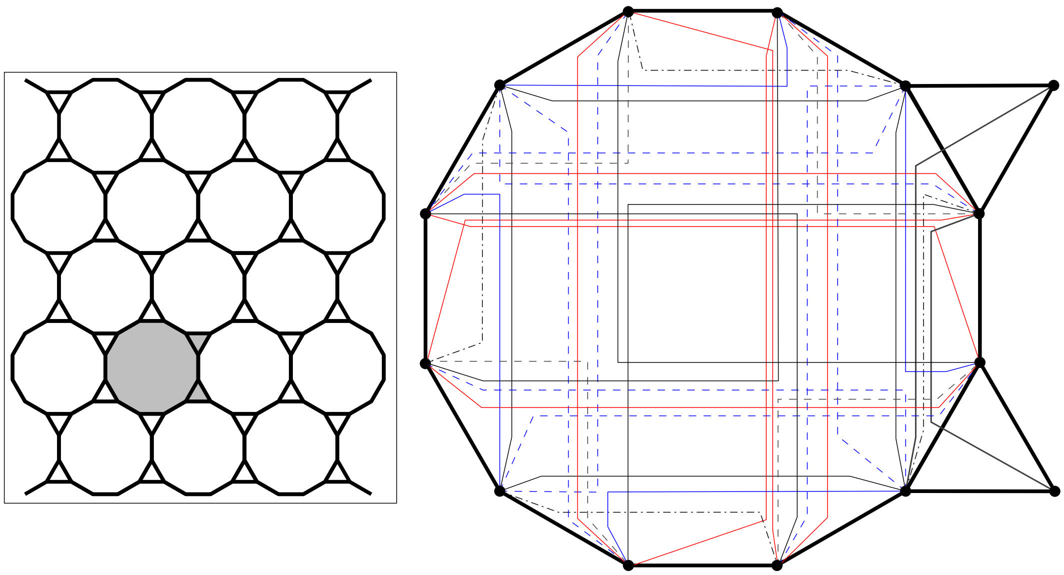

Figure 4: Lower bound construction for a RAC_2 drawing in an Archimedean tiling (12,12,3)

5 Concluding Remarks

It remains an open problem to determine the maximum number of edges of a graph with n vertices in the classes R_1 and R_2. Our upper bound in Theorem 1 may be slightly improved by refining the bound in Lemma 4. If we could strengthen the upper bound in Lemma 4 for small values of n, then (4) would improve. However, we did not pursue this direction as it would not lead to significant improvement without an extensive case analysis.

Let an $\alpha\mathrm{AC}_b^=$ drawing be a polyline drawing of a graph with b bends per edge where all crossings occur at angle *exactly* α. It is easy to show that a graph with n vertices and an $\alpha\mathrm{AC}_0^=$ drawing has at most $9n-18$ edges. The edges in each "block" can be partitioned into 3 sets of noncrossing edges, and so the graph decomposes into 3 planar graphs. Every graph admits an $\alpha\mathrm{AC}_3^=$ drawing, since every affine transformation deforms *all* crossing angles uniformly in the construction by Didimo *et al.* [7]. Very recently, Ackerman *et al.* [1] proved that every graph on n vertices that admit $\alpha\mathrm{AC}_1^=$ or $\alpha\mathrm{AC}_2^=$ drawings have $O(n)$ vertices.

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