A Tight Bound For Point Guards in Piece-Wise Convex Art Galleries

Javier Cano∗  Csaba D. Tóth†  Jorge Urrutia‡

September 18, 2010

Abstract

We consider the problem of guarding curvilinear art galleries. A closed arc $a$ joining two points, $p$ and $q$, in the plane is called a convex arc if the curve obtained by joining $a$ with the line segment $pq$ encloses a convex set. A piece-wise convex polygon $P$ with vertices $v_0, \ldots, v_{n-1}$ is the region bounded by a set $\{a_0, \ldots, a_{n-1}\}$ of $n$ convex arcs with pairwise disjoint interiors such that $a_i$ joins $v_i$ to $v_{i+1}$, addition taken mod $n$, each of them convex with respect to the interior of $P$. A piece-wise convex art gallery is the connected region bounded by a piece-wise convex polygon. We show that $\lceil \frac{n}{2} \rceil$ point guards are always sufficient in order to guard a piece-wise convex art gallery. This bound is best possible.

1 Introduction

Let $V = \{v_0, v_1, \ldots, v_{n-1}\}$ be a set of $n$ points in the plane together with a set of straight-line segments $E = \{e_0, e_1, \ldots, e_{n-1}\}$ with pairwise disjoint relative interiors such that the endpoints of $e_i$ are $v_i$ and $v_{i+1}$, $i = 0, \ldots, n-1$, addition taken mod $n$. Let $A$ be the region bounded by the closed curve obtained by joining the elements of $E$. We call $A$ an art gallery, $E$ and $V$ will be called the edge and vertex sets of $A$, respectively. A set $G \subseteq A$ of points, called guards jointly monitor $A$ if for any point $p \in A$ there is a point $q \in G$ such that the line segment $pq$ lies in $A$. A classic problem in computational geometry is finding a minimum set of guards for a given art gallery. In the 1970s, Chvátal [6] proved that $\lfloor \frac{n}{3} \rfloor$ guards are always sufficient and sometimes necessary to guard any art gallery with $n$ vertices. Since then many variations of this problem have been studied; see [18, 17, 21] for a detailed reference on art gallery problems. Applications areas of this kind of problems include robotics [12, 22], motion planning [14, 16], computer vision and pattern recognition [2, 19, 20, 23], computer graphics [5, 15], CAD/CAM [3, 7], and wireless networks [8]. Recently Karavelas, Tsigaridas, and Tóth [11] generalized the art gallery problem to curvilinear art galleries, where the edges in $E$ are arbitrary Jordan arcs with pairwise disjoint relative interiors, rather than line segments. In general, the minimum number of guards for a curvilinear art gallery cannot be bounded in terms of the number of vertices. If, however, we restrict the arcs to be convex, the number of guards is bounded by a function of the number of vertices. A Jordan arc $a_i$ between points $v_i$ and $v_{i+1}$ is convex if the closed curve containing $a_i$ and the line segment $v_i v_{i+1}$ enclose a convex region $C_i$ of the plane; see Figure 1 (left).

∗Posgrado en Ciencia e Ingeniería de la Computación, Universidad Nacional Autónoma de México, D.F. México, j_cano@uxmcc2.iimas.unam.mx
†Department of Mathematics, University of Calgary, cdtoth@math.ucalgary.ca
‡Instituto de Matemáticas, Universidad Nacional Autónoma de México, D.F. México, urrutia@matem.unam.mx
Figure 1: Left: A convex arc. Right: A piece-wise convex polygon that needs \( \lceil \frac{n}{2} \rceil \) point guards.

Notice that each convex arc has a convex and a reflex side. \( A \) is a *piece-wise convex art gallery* if its interior lies on the convex side of each arc \( a_i \); see Figure 2 (left). In the remainder of this paper, *curvilinear art gallery* will always refer to a piece-wise convex curvilinear art gallery.

Karavelas, Tsigaridas, and Tóth [11] proved that \( \lfloor \frac{2n}{3} \rfloor \) vertex guards (that is, guards restricted to be on the vertices of \( A \)) are always sufficient and sometimes necessary to guard any curvilinear art gallery with \( n \geq 2 \) vertices; they also proved that \( \lceil \frac{n}{2} \rceil \) point guards (that is, guards anywhere inside \( A \)) are sometimes necessary. Cano, Espinosa and Urrutia [4] proved that \( \lfloor \frac{5n}{8} \rfloor \) point guards are always sufficient to guard a curvilinear art gallery with \( n \geq 2 \) vertices. Recently, Karavelas [10] showed that \( \lceil \frac{2n+1}{5} \rceil \) edge guards (that is, guards allowed to move along edges of the gallery) are always sufficient to guard any curvilinear art gallery with \( n \) vertices, and \( \lceil \frac{n}{3} \rceil \) edge guards are sometimes necessary. In this paper, we prove that any curvilinear art gallery with \( n \) vertices can be guarded with at most \( \lceil \frac{n}{2} \rceil \) point guards.

**Theorem 1.** Let \( A \) be a piecewise-convex curvilinear art gallery with \( n \) vertices. Then \( \lceil \frac{n}{2} \rceil \) guards are always sufficient and sometimes necessary to guard \( A \).

The proof is based on a convex decomposition of a piece-wise convex art gallery with \( n \) vertices into \( n+1 \) convex cells. We partition the cells into \( \lceil \frac{n}{2} \rceil \) sets, each of which can be monitored by a single guard lying on their common boundary. We use a special convex decomposition (discussed in Section 3) in which every convex cell has at least two vertices of the gallery on its boundary. Such a decomposition can be constructed by a technique reminiscent of that of Al-Jubeh *et al.* [1].

It is easy to see that the upper bound of \( \lceil \frac{n}{2} \rceil \) in Theorem 1 is best possible. For every \( n \geq 3 \), there is a curvilinear art gallery with \( n \) vertices that requires \( \lceil \frac{n}{2} \rceil \) point guards. A construction due to Karavelas, Tóth and Tsigaridas [11] is shown in Figure 1 (right). To close our paper, we give a simpler proof for the sufficiency of \( \lceil \frac{2n}{3} \rceil \) vertex guards for curvilinear art galleries.

### 1.1 Notation

We introduce some notation. Let \( \sigma \) be a simple polygonal closed curve in the plane. For two vertices \( x, y \in \sigma \) we denote by \( [x, y] \) the counterclockwise path that starts with \( x \) and ends at \( y \). Clearly, \( [x, y] \cup [y, x] = \sigma \) and \( [x, y] \cap [y, x] = \{x, y\} \). Let \( P \) be a polygonal path. If the number of vertices in \( P \) is even, then we call \( P \) an *even path*, and an *odd path* otherwise. Analogously, we call a cycle with an even (resp., odd) number of vertices an *even cycle* (resp., *odd cycle*).
2 Convex Decompositions

Let $A$ be a curvilinear art gallery. A \textit{convex decomposition} of $A$ is a finite set $C$ of closed convex regions with pairwise disjoint interiors, called \textit{cells}, such that their union is $A$. The boundary of every cell in $C$ consists of some straight line segments lying in $A$ and some convex arcs contained in the boundary of $A$. We define the \textit{vertices} of $C$ to be the endpoints of the straight line segments on the boundaries of cells in $C$. Every vertex of $C$ is either a vertex of the art gallery or a \textit{Steiner vertex}, which lies in the interior of $A$ or in the relative interior of some arc $a_i$. The \textit{edges} of $C$ are the portions of these line segments between consecutive vertices of $C$. See Figure 3 for an example.

We denote by $\delta(C)$ the graph formed by the edges and vertices of $C$. We allow any possible straight line arc $a_i$ to be a (degenerate) cell in $C$, in this case $a_i$ is also an edge of $C$ lying on the boundary of the degenerate cell.

We define the \textit{dual graph} $D(C)$ of a convex decomposition $C$ as the graph whose vertices are the cells of $C$, two of which are adjacent if and only if their boundaries intersect. Observe that the cells incident to any Steiner vertex of $\delta(C)$ form a clique in $D(C)$; see Figure 2.

![Figure 2: Left: A piece-wise convex art gallery with 15 vertices. Right: The dual graph of a convex decomposition of a curvilinear art gallery.](image)

\textbf{Normal decompositions.} To prove our main result, we construct a family of convex decompositions of $A$ into $n + 1$ cells. A convex decomposition $C$ of $A$ with $n + 1$ cells is called \textit{normal} if the edges of $\delta(C)$ can be directed so that we obtain a directed graph $\vec{\delta}(C)$ satisfying the following three conditions:

1. the vertices of $A$ have out-degree 1,

2. every vertex $v$ of $C$ located in the interior of $A$ has out-degree 1,

3. every vertex of $C$ in the relative interior of an edge $a_i$ of $A$ has out-degree 0.

\textbf{Standard convex decompositions.} For a curvilinear art gallery $A$, we can easily construct a special normal decomposition in which every edge lies on a directed segment emitted by one of the vertices. For every vertex $v_i$ of $A$, let $W_i$ be the wedge formed by all rays emitted from $v_i$ that partition the (counterclockwise) angle between tangent lines to $a_i$ and $a_{i-1}$ at $v_i$ into two convex angles. (If this angle is already convex, then $W_i$ is the angular domain between the tangents of $a_{i-1}$ and $a_i$, otherwise it is between the tangents of $a_i$ and $a_{i-1}$.)
Figure 3: A standard convex decomposition (left) and a good decomposition (right) of a curvilinear art gallery.

**Lemma 2.** For every vertex \( v_i \) of \( A \), there is a directed segment \( \vec{r}_i \) lying in \( A \cap W_i \) that connects \( v_i \) to another point on the boundary of \( A \), which is not in the relative interior of arc \( a_i \) or \( a_{i+1} \).

**Proof.** We construct a directed segment \( \vec{r}_i \) for every vertex \( v_i \). We distinguish two cases. First suppose that \( C_{i-1} \) and \( C_i \) (as defined in the introduction) intersect in a single point \( v_i \). Let \( h \) be a separating line between such that \( a_{i-1} \setminus \{v_i\} \) and \( a_i \setminus \{v_i\} \) lie in two different open halfplanes bounded by \( h \). It is clear that \( h \) partitions the (counterclockwise) angle between tangent lines to \( a_{i-1} \) and \( a_i \) at \( v_i \) into two convex angles. Shoot a ray from \( v_i \) into the interior of \( A \) along \( h \), and let \( \vec{r}_i \) be the part of such a ray from \( v_i \) to the first intersection point with the boundary of \( A \).

Now suppose that \( C_{i-1} \) and \( C_i \) intersect in several points (including \( v_i \)). If \( a_{i-1} \) or \( a_i \) is a line segment, then let \( \vec{r}_i \) be this segment with a direction from \( v_i \) to \( v_{i-1} \) or \( v_{i+1} \). Now suppose that neither \( a_{i-1} \) nor \( a_i \) is a line segment. Then either the directed segment \( v_i v_{i-1} \) lies in \( C_i \) or the directed segment \( v_i v_{i+1} \) lies in \( C_{i-1} \). Let \( \vec{r}_i \) be the initial portion of this directed segment from \( v_i \) to the first intersection point with the boundary of \( A \). By construction, the endpoint of \( \vec{r}_i \) cannot be in the relative interior of arc \( a_{i-1} \) or \( a_i \).

We construct a normal decomposition for a given curvilinear art gallery as follows. For \( i = 0, 1, \ldots, n-1 \), draw a directed line segment starting from \( v_i \) along a directed segment \( \vec{r}_i \) as described in Lemma 2 until it hits the boundary of \( A \) or a previously drawn segment. See Figure 3 (left). It is clear that the \( n \) directed segments decompose \( A \) into \( n+1 \) convex cells (degenerate cells are possible if \( a_i \) or \( a_{i-1} \) is a line segment collinear with \( \vec{r}_i \)). We call any convex decomposition constructed in this way a **standard decomposition** of \( A \). Observe that the directions of segments \( \vec{r}_i \) induce a direction on all edges of \( \delta(C) \). Let \( \overrightarrow{\delta(C)} \) denote this directed graph. It is easy to verify now that every standard decomposition is a normal decomposition.

**Cyclic and acyclic cells.** Typically every cell in a normal decomposition is adjacent to the boundary of \( A \). Some cells, however, may be disjoint from the relative interior of every convex arc \( a_i \), \( i = 0, \ldots, n-1 \); see Figure 1 (right). Since the out-degree of every vertex of \( \overrightarrow{\delta(C)} \) on the boundary of such cell is one, the boundary is a directed cycle. We say that a cell in a normal decomposition is **cyclic** if it is disjoint from the relative interior of every arc \( a_i \), \( i = 0, \ldots, n-1 \), and **acyclic** otherwise.

**Good and bad cells.** Let \( C \) be a normal convex decomposition of \( A \). A cell \( c \) of \( C \) is called **good** if its boundary contains at least two vertices of \( A \), otherwise \( c \) is called **bad**. A convex decomposition
of a curvilinear polygon is called good if all of its cells are good; see Figure 3 (right). The following observation about standard convex decompositions will be useful.

Observation 1. Every cell of a standard convex decomposition of $A$ contains at least one vertex of $A$ on its boundary.

Proof. Let $c$ be a cell in a standard convex decomposition $C$. Let $i$, $0 \leq i \leq n - 1$, be the largest index such that the boundary of $c$ contains some portion of the line segment starting from $v_i$. Since no other edge of $c$ can hit the relative interior of this segment, the segment endpoint $v_i$ also lies on the boundary of $c$.

We can now classify the components of $\vec{δ}(C)$.

Lemma 3. Let $C$ be a normal decomposition of $A$. Then every connected component of $\vec{δ}(C)$ is either a directed tree rooted at a point in the relative interior of an edge of $A$, or a directed graph with exactly one directed cycle which bounds a cyclic cell.

Proof. Consider a connected component $t$ of $\vec{δ}(C)$. Since the out-degree of every vertex of $\vec{δ}(C)$ is at most one, $t$ contains at most one directed cycle. If $t$ contains no cycle, then it is a rooted tree, and the root has to be a point with out-degree 0; that is, a Steiner point lying in the relative interior of an edge of $A$. Now suppose that $t$ contains a cycle, say $σ$. Note that the interior of $σ$ lies in the interior of $A$, since $A$ is simply connected. It remains to show that $σ$ bounds a single cell in $C$. Suppose, to the contrary, that at least two cells of $C$ are inside $σ$. These cells must be separated by some edges of $δ(C)$ which are not part of $σ$. None of these edges can start from a vertex of $σ$, otherwise the out-degree restriction is not satisfied. Hence, at least one of these edges has to start from a vertex of $A$. This, however, is impossible since the interior of $σ$ lies in the interior of $A$. We conclude that the interior of $σ$ is a single cell in $C$.

Special cells for each component of $\vec{δ}(C)$. Let $t$ be a connected component of $\vec{δ}(C)$. We say that a cell $c \in C$ is incident to $t$ if the boundary of $c$ contains at least one edge of $t$. We specify some special cells for $t$. If $t$ contains a cycle $σ$, then let the cell bounded by $σ$ be special. If $t$ is a directed tree rooted at some vertex $x$ (lying in the relative interior of some arc $a_i$), then let the two cells incident to $x$ having an arc of $a_i$ on its boundary be special.

3 Constructing a Good Normal Decomposition

In this section we construct a good normal decomposition for a curvilinear art gallery with $n \geq 3$ vertices.

Lemma 4. Every curvilinear art gallery with $n \geq 3$ vertices has a good normal decomposition.

Proof. Let $C$ be a standard convex decomposition of $A$. If $C$ is a good decomposition, then our proof is complete. Otherwise we will deform $\vec{δ}(C)$ continuously into a good decomposition. Our algorithm successively processes every bad cell of $C$, deforming its boundary until it contains at least two vertices of $A$. During the deformation, we maintain a normal decomposition and good cells remain good. Specifically, we maintain the following four invariants:

I1 $C$ is a normal decomposition of $A$.

I2 For every edge $e$ of $\vec{δ}(C)$, there is a vertex $v$ of $A$ such that $\vec{δ}(C)$ contains a directed path of collinear edges, including $e$, that either starts from $v$ or ends at $v$.
I3 If a cell $c \in C$ is incident to a vertex $v$ of $A$, then $c$ remains incident to $v$.

I4 If a cell $c \in C$ is cyclic, then it remains cyclic.

Note that Invariants I1 and I2 hold for every standard convex decomposition. Invariant I3 implies that when all bad cells have been processed, we obtain a good decomposition of $A$.

Consider a bad cell $c$ of $C$. We process cell $c$ while maintaining invariants I1–I4. We first process all acyclic bad cells and then process all cyclic bad cells as follows.

**Processing an acyclic bad cell.** Let $c \in C$ be an acyclic bad cell. By Observation 1 and invariant I3, the boundary of $c$ contains exactly one vertex of $A$, which we denote by $v_i$. The edges of $\bar{\delta}(C)$ on the boundary of $c$ induce a directed path $\pi$ in $\bar{\delta}(C)$ which starts at vertex $v_i$. Since $c$ is acyclic, $\pi$ ends at a point $x$ in the relative interior of an edge $a$ of $A$ adjacent to $v_i$. Without loss of generality, we may assume that $a = a_i$, and thus $v_{i+1}$ is the other endpoint of $a_i$. Refer to Fig. 4.

![Figure 4: Stretching segment $\bar{yx}$.](image)

Observe that some edges along $\pi$ may be collinear. Let $e_1, e_2, \ldots, e_k$ be the maximal directed line segments that contain collinear edges of $\pi$ in this order such that $e_1$ starts from $v_i$ and $e_k$ ends at $x$. Let $\bar{yx} = e_k$. We process $c$ as follows. Move point $x$ continuously along $a_i$ towards $v_{i+1}$ and stretch the directed edge $\bar{yx}$ until one of the following possibilities arises:

1. We have $k \geq 2$ and $\bar{yx}$ becomes collinear with $e_{k-1}$. Then set $k := k - 1$, recompute $y$, and continue moving $x$ (see Figure 5).

2. We have $x = v_{i+1}$ (see Figure 6, left) or some vertex $v_r$ of $A$ appears in the relative interior of $\bar{yx}$ (see Figure 6, right).

![Figure 5: $\bar{yx}$ becomes collinear with $e_{k-1}$](image)

While stretching segment $\bar{yx}$, the edges of $\bar{\delta}(C)$ that hit $\bar{yx}$ from the opposite side of $c$ are continuously shortened, and the edges and Steiner vertices completely swept by $\bar{yx}$ disappear. If at the
beginning of processing cell $c$, point $x$ is adjacent to another bad acyclic cell $c'$, and the outgoing edge of $v_{i+1}$ is shortened to a single point when we move $x$ to $v_{i+1}$, then add a new directed edge $v_{i+1}v_i'$ (which effectively decomposes cell $c$ into two good cells). This completes the description of the processing of cell $c$. The process terminates, since at each step, either $k$ is decremented or $c$ becomes a good cell.

We show next that invariants I1–I4 are maintained. First we show that $c$ remains convex. The first stopping rule guarantees that $c$ has convex angles at every internal vertex of path $\pi$. If $y = v_i$, then cell $c$ remains convex at $v_i$, since $\overrightarrow{yx}$ connects two points of the convex arc $a_i$. It is clear that the cells on the opposite side of $\overrightarrow{yx}$ remain convex. The only case when a cell $c'$ can disappear is when $c'$ is a bad cell incident to $v_{i+1}$, we move $x$ to $v_{i+1}$, and the outgoing edge of $v_{i+1}$ is shortened to a single point. In this case, however, we add a new outgoing edge at $v_{i+1}$, and split $c$ into two good cells, thereby restoring a normal decomposition. Invariant I2 continues to hold for all edges of $\overrightarrow{yx}$ that we do not modify. The edges along $\overrightarrow{yx}$ do not satisfy I2 during the continuous motion. At the end of the process, $\overrightarrow{yx}$ contains a vertex of $A$, and so I2 becomes true for all edges along $\overrightarrow{yx}$. It is easy to verify that invariants I3 and I4 are maintained.

![Figure 6: $yx$ hits a vertex of $A$.](image)

Also observe that if $\ell$ hits $v_{i+1}$, a cell $c'$ on the opposite side of $\overrightarrow{yx}$ may become cyclic, see Figure 6 (left).

**Processing a cyclic bad cell.** Let $c$ be a cyclic bad cell of $C$. The boundary of $c$ is a directed cycle $\pi$ in $\overrightarrow{\delta}(C)$. Refer to Fig. 7. By Observation 1 and invariant I3, its boundary contains some vertex $v_i$ of $A$. Some edges along $\pi$ may be collinear. Let $e_1, e_2, \ldots, e_k$ be the maximal directed line segments that contain collinear edges of $\pi$ in this order such that $e_1$ starts from $v_i$ and $e_k$ ends at $v_i$. Note that $k \geq 3$, and let $\overrightarrow{yx} = e_{k-2}$. We process $c$ as follows. By invariant I2, $\overrightarrow{\delta}(C)$ contains a directed path through $e_{k-1}$ and starting or ending at some vertex $w$ of $A$. Since the directed path passing through $e_{k-1}$ bends at the endpoint of $e_{k-1}$, there is a collinear directed path from $w$ through $e_{k-1}$, including point $x$. Let $\ell = \overrightarrow{wx} \subset \overrightarrow{\delta}(C)$. Move point $x$ continuously along $\ell$ towards $w$ and stretch the directed edge $\overrightarrow{yx}$ until one of the following possibilities arises:

1. We have $k \geq 4$ and $\overrightarrow{yx}$ becomes collinear with $e_{k-3}$. Then set $k := k - 1$, recompute $y$, and continue moving $x$.

2. We have $x = w$ (see Figure 7, right) or some vertex $v_r$ of $A$ appears in the relative interior of $\overrightarrow{yx}$.

This completes the description of the processing of a cyclic cell $c$. The process terminates, since at each step, either $k$ is decremented or $c$ becomes a good cell.

We show next that invariants I1–I4 are maintained. The first stopping rule guarantees that $c$ has convex angles at every vertex of cycle $\pi$. It is clear that the cells on the opposite side of $\overrightarrow{yx}$ remain
convex, and cannot disappear. Invariant I2 continue to hold for all edges of $\vec{\delta}(C)$ that we do not modify. Similar to the processing of acyclic cells, the edges along $\vec{yx}$ do not satisfy I2 during the continuous motion. At the end of the process, $\vec{yx}$ contains a vertex of $A$, and so I2 becomes true for all edges along $\vec{yx}$. It is easy to verify that invariants I3 and I4 are maintained.

4 The Dual Graphs of Good Normal Decompositions

Let $C$ be a good normal decomposition of $A$. Since $C$ is fixed, we will refer to $\vec{\delta}(C)$ simply as $\vec{\delta}$. Let $t$ be a connected component of $\vec{\delta}$ adjacent to at least three cells of $C$. Let $D(t)$ be the subgraph of $D(C)$ induced by the cells of $C$ incident to $t$. In this section, we prove several important properties of $D(t)$. We begin with an easy observation.

Observation 2. Every vertex of $D(C)$ has degree at least 2.

Proof. Let $c \in C$ be a convex cell. Clearly the degree of every vertex is at least the number of edges of $\vec{\delta}$ on its boundary, and every cell is adjacent to at least one edge of $\vec{\delta}$. Suppose that $c$ is adjacent to exactly one edge $e$ of $\vec{\delta}$. Since $c$ is good, both endpoints of $e$ are vertices of $A$. Let $\vec{v} = \vec{w}$. Since $v$ has out-degree 1, there is another edge, say $e'$, that starts from $v$, and lies between some cells $c_1$ and $c_2$. Since $v$ is on the boundary of $c$, it is adjacent to both $c_1$ and $c_2$ in $D(C)$. □

The following lemmas are the key to our result.

Lemma 5. $D(t)$ contains a cycle that passes through all acyclic cells adjacent to $t$.

Proof. Recall that by Lemma 3, $t$ is either a rooted tree or it contains a directed cycle bounding a cyclic cell of $C$.

We construct a cycle $H_a(t)$ in $D(t)$ as follows. Walk around the boundary of $A$ starting from an arbitrary point. We say that the walk encounters a cell $c$ if the walk traverses an arc on the boundary of $c$ (rather than either passing through only one vertex on the boundary of $c$ or none at all). Relabel the cells represented by vertices of $D(t)$ along the boundary of $A$ to $c_1, \ldots, c_k$ in the order in which they are encountered in this walk. The order is well defined: if the walk encounters a cell $c_i$ twice, say at arcs $\gamma_1$ and $\gamma_2$, then the portion of the boundary of $A$ between $\gamma_1$ and $\gamma_2$ is separated from $t$ by cell $c_i$, and cannot encounter any other cell adjacent to $t$. Let $H_a(t) = (c_1, \ldots, c_k)$. It is clear that consecutive cells in $H_a(t)$ are adjacent in $D(t)$. That is, $H_a(t)$ is a simple cycle in $D(t)$, which passes through all acyclic cells adjacent to $t$, as required. □

Lemma 6. Let $c$ be a special cell adjacent to $t$, and assume that $c$ is not adjacent to any other component of $\vec{\delta}$. Then there is a vertex $v(c)$ of $A$ incident to $c$ such that $v(c)$ is incident to two more cells $c_1, c_2 \in C \setminus \{c\}$ which are consecutive in $H_a(t)$.
Proof. Suppose first that $t$ is a directed tree (see Figure 8). Since $c$ is special, the root $x$ of $t$ is on the boundary of $c$. Suppose that the part of $t$ lying on the boundary of $c$ is the directed path $\pi$ from vertex $v_1$ to $x$. Since $c$ is not adjacent to any other component of $\delta$, the root $x$ lies on a convex arc of $A$ incident to $v_i$. However, cell $c$ is good, and so the boundary of $c$ contains at least one more vertex of $A$, $v(c)$, which is an internal vertex of path $\pi$. Since $v(c)$ is an internal vertex of $\pi$, there are two cells, say $c_1, c_2 \in C \setminus \{c\}$, whose boundaries each contain $v(c)$ and some initial part of a convex arc incident to $v(c)$. By construction, $c_1$ and $c_2$ are adjacent in the cycle $H_a(t)$. Now suppose that $c$ is a cyclic cell, bounded by a cycle $\sigma$ of $t$. Let $v(c)$ be an arbitrary vertex of $A$ along $\sigma$. Let $c_1, c_2 \in C \setminus \{c\}$ be the cells whose boundaries each contain $v(c)$ and some initial part of a convex arc incident to $v(c)$. Again, $c_1$ and $c_2$ are adjacent in the cycle $H_a(t)$. \hfill \Box

![Figure 8: Two cells $c_1, c_2$, adjacent to $c$ in $D(t)$.](image)

**Corollary 7.** $D(t)$ is Hamiltonian.

*Proof.* If $t$ is a directed tree, then $H_a(t)$ is a Hamiltonian cycle of $D(t)$ by Lemma 5. If $t$ has a cycle, then cycle $H_a(t)$ passes through all acyclic cells, but misses one cyclic cell $c$. By Lemma 6 there are two consecutive cells, $c_1$ and $c_2$, in $H_a(t)$ that are both adjacent to $c$ in $D(t)$. By removing the edge $c_1c_2$ from $H_a(t)$ and connecting $c$ with $c_1$ and $c_2$ we obtain a Hamiltonian cycle in $D(t)$.

In the remainder of this paper, we denote by $H(t)$ the Hamiltonian cycle constructed in the proof of Corollary 7.

**Basic cycles.** Let $\gamma$ be a simple cycle in graph $D(t)$. We define region $R_\gamma$ in the plane as the union of the cells in $\gamma$. Observe that region $R_\gamma$ is simply connected if and only if the cells in $\gamma$ do not enclose any cyclic cell $c \notin \gamma$. We say that $\gamma$ is a basic cycle of $D(t)$ if $R_\gamma$ is simply connected; see Figure 9 (left). In particular, $H(t)$ is a basic cycle, and if $t$ is a tree, then every simple cycle in $D(t)$ is basic. We denote by $D(t, \gamma)$ the subgraph of $D(t)$ induced by the vertices of $\gamma$.

**Lemma 8.** Every basic cycle $\gamma$ in $D(t)$ with $k \geq 3$ cells contains three consecutive cells incident to a vertex of $\delta$.

*Proof.* Label the cells in $\gamma$ counterclockwise by $c_0, c_1, \ldots, c_{k-1}$ along the boundary of $R_\gamma$. If $c_i$ and $c_j$, $i + 1 < j$, are adjacent in $D(t)$, then $\gamma' = (c_i, c_{i+1}, \ldots, c_j)$ is called a sub-cycle of $\gamma$, addition taken mod $k$, Figure 9 (right). Every sub-cycle $\gamma'$ is a basic cycle, since $R_{\gamma'} \subset R_\gamma$ contains no cell in its interior. It is enough to show that $\gamma$ has a sub-cycle $\gamma'$ of 3 cells: the common boundary between the three consecutive cells in $\gamma'$ meets, since $R_\gamma$ is simply connected.
Let $\gamma'$ be the smallest sub-cycle of $\gamma$. By relabelling the cells if necessary, we may assume that $\gamma' = (c_0, c_1, \ldots, c_i)$. If $i = 2$, then our proof is complete. Assume that $i \geq 3$. The boundary between $c_0$ and $c_1$ is a (possibly degenerate) line segment $s$. Since $R_\gamma$ is simply connected, one endpoint of $s$ must be incident to some other cell $c_j$ in $\gamma'$. Hence $\gamma'' = (c_0, c_1, \ldots, c_j)$ is a strictly smaller sub-cycle of $\gamma$, contradicting the minimality of $\gamma'$.

**Lemma 9.** For every basic cycle $\gamma$ in $D(t)$ with $k \geq 3$ cells, graph $D(t, \gamma)$ has a clique cover of size $\lceil k/2 \rceil$ such that the cells in each clique can be guarded from a single point.

**Proof.** If $\gamma$ is an even cycle, then it has a perfect matching of size $\lceil k/2 \rceil$, which is a desired clique cover, so we are done. Suppose that $\gamma$ is odd. By Lemma 8, $\gamma$ contains three consecutive cells incident to a common vertex of $\vec{\delta}$. This triple together with a perfect matching on the remaining $k - 3$ vertices of $\gamma$ is a desired clique cover of size $\lceil k/2 \rceil$.

**5 Constructing a Guard Set**

**Proof of Theorem 1.** Let $A$ be a curvilinear art gallery with $n \geq 3$ vertices. Fix a good normal decomposition $C$ of $A$. As we noted before, each vertex of $\vec{\delta}$ corresponds to a clique in the dual graph $D(C)$. To show that $A$ can be guarded by at most $\lceil n/2 \rceil$ point guards, it is enough to show that $D(C)$ has a clique cover of size at most $\lceil n/2 \rceil$ such that each clique is induced by some vertex of $\vec{\delta}$, and so the convex cells in each of these cliques can be guarded from a single point. In the remainder of the proof we describe an algorithm for constructing a clique cover of $D(C)$ having this property and size at most $\lceil n/2 \rceil$.

We define a graph $\Gamma$ on the connected components of $\vec{\delta}$. Two connected components $t$ and $t'$ of $\vec{\delta}$ are adjacent in $\Gamma$ if and only if there is a cell $c \in C$ adjacent to both of them; see Figure 10. Notice that all the components of $\vec{\delta}$ incident to a cell $c \in C$ induce a clique in $\Gamma$. Relabel the components of $\vec{\delta}$ by $t_1, \ldots, t_k$ according to a breadth-first search traversal of $\Gamma$. Notice that this labelling has the property that every $t_m$ is adjacent to at most one cell which is adjacent to some previous component $t_i$ with $i < m$; otherwise $A$ would not be simply connected. Let $n(t_m)$ be the number of cells in $D(t_m)$. Note that $n(t_m) > 1$.

We construct a clique cover $\mathcal{G}$ of $D(C)$ as follows. Initially, let $\mathcal{G} = \emptyset$. Our algorithm runs in $k$ iterations. In iteration $m = 1, 2, \ldots, k$, we process graph $D(t_m)$ and compute a set $\mathcal{G}_m$ such that the cliques in $\bigcup_{i=1}^m \mathcal{G}_i$ cover all but at most one cells in $D(t_m)$. We may leave at most one cell in $D(t_m)$ uncovered provided that it is contained in $D(t_j)$ for some $j > m$ (which will be processed later).
Recall that for every $t_m$, at most one cell of $D(t_m)$ is contained in a previously processed $D(t_i)$, $i < m$. This cell may or may not be covered by a clique in $G_m$. Accordingly, at the beginning of the $m$-th iteration two cases may arise:

**Case a: No cell of $D(t_m)$ has been covered in any previous iteration.** We proceed as follows: If $n(t_m) = 2$, then $D(t_m)$ is a clique of size $1 = n(t_m)/2$. If $t_m \geq 3$, by Corollary 7 and Lemma 9, $D(t_m)$ has a desired clique cover of size $\left\lceil \frac{n(t_m)}{2} \right\rceil$.

**Case b: Exactly one cell of $D(t_m)$, say $c$, was covered in a previous iteration.** If $n(t_m) = 2$, then let $G_m = \emptyset$. Then one cell in $D(t_m)$ is still uncovered. By Observation 2, the uncovered cell is adjacent to some other component $t_j$ with $j > m$, which will be processed later. In the remainder of the proof, we assume $n(t_m) \geq 3$. Suppose that cell $c \in D(t_m)$ is already covered. We will distinguish several subcases. In each subcase, we partition $D(t_m) \setminus \{c\}$ into subgraphs that are cliques induced by a vertex of $\vec{c}$, even paths, basic cycles, and at most one singleton (a cell adjacent to a component $t_j$, $j > m$). A perfect matching of an even path of length $\ell$ is a clique cover of size $\ell/2$. By Lemma 9, a basic cycle of size $\ell$ has a clique cover of size $\lceil \ell/2 \rceil$. This guarantees that we obtain a desired clique cover $G_m$ of size at most $\left\lceil \frac{n(t_m) - 1}{2} \right\rceil$. We continue with the details. If $n(t_m)$ is odd, then $H(t_m) \setminus \{c\}$ is an even path. If $n(t_m)$ is even, then several sub-cases arise depending on whether $D(t_m)$ has a cyclic cell or not.

**Case b1: $D(t_m)$ has a cyclic cell $c_1 \in D(t_m)$.** Note that $c \neq c_1$, since the cyclic cell is adjacent to $t_m$ only. Since $c_1$ is a good cell, its boundary contains at least two vertices of $A$. By Lemma 6, each vertex of $A$ on the boundary of $c_1$ is incident to two consecutive cells in the cycle $H_a(t_m)$. Therefore there are two pairs of consecutive vertices, $c_2, c_3$ and $c_4, c_5$ in counterclockwise order along $H_a(t_m)$ (with possibly $c_3 = c_4$ or $c_2 = c_5$) such that $\{c_1, c_2, c_3\}$ and $\{c_1, c_4, c_5\}$ are cliques, each of which can be guarded from a vertex of $A$. See Figure 11.

Partition the cycle $H_a(t_m)$ into paths $[c_3, c_4]$ and $[c_5, c_2]$. Suppose without loss of generality that $c \in [c_5, c_2]$. Clearly $[c_5, c_2] \setminus \{c\}$ is the union of two (possibly empty) paths, which we denote by $p_2$ and $p_5$ such that $c_2 \in p_2$ and $c_5 \in p_5$ respectively. Note that either $p_2$ or $p_2 \setminus \{c_2\}$ is even; denote this path by $p'_2$. Similarly either $p_5$ or $p_5 \setminus \{c_5\}$ is even, and denoted by $p'_5$. Since $c_1$ is adjacent to $c_2, c_3, c_4, c_5$, the graph $D(t_m) \setminus (\{c\} \cup p'_2 \cup p'_5)$ has a spanning cycle, that contains $c_1$, and so it is a basic cycle.

![Figure 10: A good normal decomposition of a curvilinear art gallery, and the corresponding graph $\Gamma$.](image)
Case b2: *D(t_m) has no cyclic cell.* Let c₁ and c₂ be the special cells adjacent to t_m. We distinguish three subcases depending on whether c₁ and c₂ are adjacent to any other component of \( \vec{\delta} \) or we can apply Lemma 6:

**Case b2.1:** Both c₁ and c₂ are adjacent to some other components of \( \vec{\delta} \). Recall that \( H(t_m) \) is a Hamiltonian cycle of \( D(t_m) \) in which c₁ and c₂ are consecutive cells. Since \( n(t_m) \) is even, \( H(t_m) \setminus \{c\} \) is an odd path. First suppose that c is a special cell of \( t_m \), say \( c = c_1 \). Then \( H(t_m)\setminus \{c_1,c_2\} \) is an even path, and we leave c₂ uncovered. Now suppose that c is not a special cell of \( t_m \). Then \( H(t_m)\setminus \{c,c_1\} \) or \( H(t_m)\setminus \{c,c_2\} \) is the union of two even paths. Suppose without loss of generality that this happens for \( H(t_m)\setminus \{c,c_1\} \), and we leave c₁ uncovered.

**Case b2.2:** Exactly one of c₁ or c₂ is adjacent to some other component of \( \vec{\delta} \). Assume without loss of generality that c₁ is adjacent to no other component of \( \vec{\delta} \). By Lemma 6, there are two consecutive cells, c₃ and c₄, along \( H(t_m) \) such that \( \{c_1,c_3,c_4\} \) is a clique which can be guarded from a single point. The edge c₃c₄ splits the cycle \( H(t_m) \) into two cycles, which we denote by say \( H_1 \) and \( H_2 \) respectively such that \( H_1 \cap H_2 = \{c_1,c_3\} \). We may assume without loss of generality that \( c_4 \in H_1 \) and \( c_2 \in H_2 \); see Figure 12. Now we have:

**Case b2.2.1:** c \( \in H_1 \). Then the cells of \( D(t_m) \setminus \{c,c_2\} \) lie on two paths: \( p_1 = H_1 \setminus \{c\} \) and \( p_2 = H_2 \setminus \{c_1,c_2,c_3\} \). See Figure 12(a). We leave c₂ uncovered. If \( p_1 \) is even, then \( p_2 \) is even, too, and we have a desired partition of \( D(t_m) \setminus \{c\} \). If \( p_1 \) is odd, then \( p_2 \) is odd, too. By construction, edge \( c_4c_1 \) is a triangular chord of \( p_1 \). We obtain an even path \( p'_1 \) from \( p_1 \) by replacing edges \( c_1c_3 \) and \( c_3c_4 \) with the edge \( c_1c_4 \). We obtain an even path \( p'_2 \) from \( p_2 \) by appending \( c_3 \) to it.
**Case b2.2.2:** \( c \in H_2 \). Then \( H_2 \setminus \{c, c_2\} \) is the union of two (possibly empty) paths, which we denote by \( p_2 \) and \( p_3 \) such that \( c_2 \in p_2 \) and \( c_3 \in p_3 \) respectively; see Figure 12(b). Note that either \( p_2 \) or \( p_2 \setminus \{c_2\} \) is even; denote this path by \( p'_2 \). Similarly either \( p_3 \) or \( p_3 \setminus \{c_3\} \) is even, and denoted by \( p'_3 \). Since \( c_1 \) is adjacent to \( c_3 \) and \( c_4 \), the graph \( H_1 \setminus p'_3 \) has a spanning cycle \( H'_1 \) which is a basic cycle. The even paths \( p'_2 \) and \( p'_3 \), basic cycle \( H'_1 \), and possibly leaving cell \( c_2 \) as a singleton, we have a desired partition of \( D(t_m) \setminus \{c\} \).

---

**Case b2.3:** Neither \( c_1 \) nor \( c_2 \) is adjacent to any other component of \( \vec{\delta} \). By Lemma 6, there are two consecutive cells, \( c_3 \) and \( c_4 \), along \( H(t_m) \) such that \( \{c_2, c_3, c_4\} \) is a clique which can be guarded from a single vertex \( v(c_2) \). Similarly, there are two consecutive cells, \( c_5 \) and \( c_6 \), along \( H(t_m) \) such that \( \{c_1, c_5, c_6\} \) is a clique which can be guarded from a single vertex \( v(c_1) \). We distinguish two subcases depending on whether the vertices \( v(c_1) \) and \( v(c_2) \) are distinct:

---

**Case b2.3.1:** \( v(c_1) \neq v(c_2) \). Suppose without loss of generality that \( c_3, c_4, c_5, c_6 \) are in counterclockwise order, with possibly \( c_4 = c_5 \); see Figure 13. Let \( p_1 = [c_4, c_5] \) along \( H(t_m) \). Let \( H_1 = [c_5, c_1] \cup c_1 c_5 \) and \( H_2 = [c_2, c_4] \cup c_4 c_2 \) be two interior disjoint cycles of \( D(t_m) \); see Figure 13.

Suppose first that \( c \in H_1 \) (we can argue analogously if \( c \in H_2 \)). Let \( p_2 = H_1 \setminus \{c\} \) be a path. We partition \( D(t_m) \setminus \{c\} \) into two even paths and a basic cycle. If \( p_2 \) is even, then set \( p_1 = p_1 \setminus \{c_3\} \), otherwise set \( p_2 = (p_2 \setminus \{c_3\}) \cup c_1 c_4 \). If \( p_1 \) is even, then set \( H_2 = H_2 \setminus \{c_5\} \cup c_5 c_6 \), otherwise set \( p_1 = p_1 \setminus \{c_5\} \). We have partitioned \( D(t_m) \setminus \{c\} \) into the even paths \( p_1 \) and \( p_2 \) and basic cycle \( H_2 \). Suppose next that \( c \in p_1 \). Now \( p_1 \setminus \{c\} \) is the union of two paths, say \( p_4 \) and \( p_5 \), such that \( c_4 \in p_4 \) and \( c_5 \in p_5 \). As in the above, depending on the parity of \( p_4 \) and \( p_5 \), we can choose to remove \( c_4 \) from \( p_4 \) or from \( H_2 \), and similarly remove \( c_5 \) from \( p_5 \) or \( H_1 \), obtaining two even paths and two basic cycles.

---

**Case b2.3.2:** \( v(c_1) = v(c_2) \). This implies that \( c_3 = c_5 \) and \( c_4 = c_6 \), and \( c_1, c_2, c_3, c_4 \) induce a 4-clique, whose vertices can be guarded from vertex \( v(c_1) = v(c_2) \). Denote the 4-clique by \( q \).

Let \( H_1 = [c_4, c_1] \cup c_1 c_4 \) and \( H_2 = [c_2, c_3] \cup c_3 c_2 \) be two cycles of \( D(t_m) \); see Figure 14a. Assume that \( c \in H_1 \) (we can argue analogously if \( c \in H_2 \)). Let \( p_1 = H_1 \setminus \{c\} \). If \( p_1 \) is even, then \( p_1 \) and \( H_2 \) is the desired partition of \( D(t_m) \setminus \{c\} \). So suppose that \( p_1 \) is odd. Notice that \( H(t_m) \setminus \{c, c_1, c_2, c_3, c_4\} \) is the union of three paths, two of which are even and the remaining path is odd. Note that one endpoint of each path is adjacent to a cell in clique \( q \). We can append one cell of \( q \) to the odd path, and obtain a partition of \( D(t_m) \setminus \{c\} \) into three even paths and a triangle contained in \( q \).
For $m = 1, 2, \ldots, k$, we have computed a set $G_m$ such that $G = \bigcup_{m=1}^{k} G_m$ is a clique cover of $D(C)$. We have $|G_m| \leq \left\lfloor \frac{n(t_m)}{2} \right\rfloor$ for every $m$. Recall that for every component $t_m$, at most one adjacent cell could be adjacent to another a previous component $t_i, i < m$. It follows that $|G| \leq \left\lfloor \frac{n+1}{2} \right\rfloor = \left\lceil \frac{n}{2} \right\rceil$.

6 A Simpler Proof for Vertex Guards

Karavelas, Tóth and Tsigaridas [11] proved that $\left\lfloor \frac{2n}{3} \right\rfloor$ vertex guards are always sufficient and sometimes necessary to guard a piece-wise convex curvilinear polygon with $n \geq 2$ vertices. We finish this paper by providing a simpler proof of their result.

**Theorem 10** ([11]). Let $A$ be a piece-wise convex curvilinear art gallery with $n \geq 2$ vertices. Then $\left\lfloor \frac{2n}{3} \right\rfloor$ vertex guards are always sufficient and sometimes necessary to guard $A$.

**Proof.** Let $A$ be a curvilinear art gallery with $n \geq 2$ vertices. Label the vertices by $v_0, \ldots, v_{n-1}$ along the boundary of $A$, addition taken mod $n$. For any two consecutive vertices $v_i$ and $v_{i+1}$, let $P_i$ be the shortest path from $v_i$ to $v_{i+1}$ contained in $A$. Refer to Figure 15. Every path $P_i$ is a simple polygonal chain. Since the boundary of $A$ consists of convex arcs, every vertex of $P_i$ is a vertex of $A$. Since $P_i$ and the convex arc $a_i$ have the same endpoints, $v_i$ and $v_{i+1}$, $P_i \cup a_i$ is a simple closed curve. Let $R_i$ denote the simply connected region in the interior of $P_i \cup a_i$. We call $R_i$ the room of $a_i$. Since $P_i$ is a shortest path between $v_i$ and $v_{i+1}$ in $A$, all internal vertices of $P_i$ are reflex vertices of region $R_i$.

Since the paths $P_i$ connect consecutive vertices of $A$, they are pairwise non-crossing, and the rooms $R_i$ are interior disjoint. The paths $P_i, i = 0, 1, \ldots, n-1$, jointly decompose $A$ into simply connected regions, see Figure 15 (right). The regions adjacent to the boundary of $A$ are rooms. We call any other region a polygonal region; these are simple polygons bounded by some edges of a path $P_i$.

Let $V$ be the set of $n$ vertices of $A$. Consider the decomposition of $A$ into $n$ rooms and possibly some polygonal regions. Triangulate every polygonal region and let $E$ denote the set of edges of all paths $P_i$, and all edges of the triangulations of the polygonal regions. We define a dual graph $T$ of graph $(V, E)$ as follows. The vertices of $T$ are the triangles in the triangulation of the polygonal regions. Two nodes are adjacent if and only if the corresponding triangles share an edge; that is, if each edge of the dual graph $T$ corresponds to an edge $e \in E$.

It is not difficult to see that $T$ is a forest. Every edge $e \in E$ decomposes $A$ into two curvilinear art galleries, and so the removal of the dual edge of $e$ disconnects one of the connected components of $T$. It follows that graph $(V, E)$ has a proper 3-vertex coloring. Fix an arbitrary 3-vertex coloring of $(V, E)$; see Figure 15 (left). The total size of the two smallest color classes is at most $\left\lfloor \frac{2n}{3} \right\rfloor$. We show that guards at these vertices jointly monitor the entire art gallery. It is clear that every

![Figure 14: Illustration for Case b2.3.2.](image-url)
triangle in the triangulation of a polygonal region is guarded by vertices in each color class. We show next that every room is guarded by vertices in any two color classes. We say that a point \( p \in A \) sees all of an edge \( e \in E \) if the triangle spanned by \( e \) and \( p \) is contained in \( A \). The following claim implies that every point in a room sees both endpoints of some edge in \( E \).

**Claim.** Let \( R_i \) be a room of \( A \), and let \( p \in R_i \). Then \( p \) sees all of some edge \( e \) in path \( P_i \).

If \( P_i \) has exactly one edge \( e \), then the room \( R_i \) is convex, and \( p \) sees all of \( e \). Suppose that \( P_i \) has at least two edges. Suppose that \( P_i = (v_i = u_0, u_1, \ldots, u_k = v_{i+1}) \). For \( j = 1, \ldots, k - 1 \), extend edge \( u_{j-1}u_j \) beyond its endpoint \( u_j \) until it hits the convex arc \( a_i \). The extensions decompose \( R_i \) into \( k - 1 \) convex cells, each adjacent to a unique edge of \( P_i \). If \( p \) lies in the interior of a convex cell, then \( p \) sees all of the edge of \( P_i \) adjacent to the cell. If \( p \) lies on the extension of edge \( u_{j-1}u_j \) for some \( j = 1, 2, \ldots, k - 1 \), then \( p \) sees all of edge \( u_ju_{j+1} \). This completes the proof of the Claim, and thus the proof of the theorem.

We conclude by constructing a family of curvilinear art galleries with \( n \) vertices, where \( n \equiv 0 \mod 3 \), that requires at least \( \frac{2n}{3} \) vertex guards. A similar construction has been presented in [11]. The construction is based on a pattern formed by three consecutive convex arcs depicted in Figure 17 (left). Let \( Q \) be a regular \( \frac{2n}{3} \)-gon, replace every edge of \( Q \) by a rotated copy of the three convex arcs as shown in Figure 17 (right). For each triple of consecutive arcs, we can construct three interior-disjoint regions such that each region is seen from only two vertices of the pattern. It now follows that the three regions require at least two vertex guards. Over \( \frac{2n}{3} \) copies of this pattern, \( n \) interior disjoint regions require \( \frac{2n}{3} \) vertex guards.
Figure 17: Left: Basic pattern for the lower bound construction. Right: A curvilinear art gallery with 27 vertices that requires 18 vertex guards.

References


