# Convex Subdivisions with Low Stabbing Numbers

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**Abstract**

It is shown that for every subdivision of the $d$-dimensional Euclidean space, $d \geq 2$, into $n$ convex cells, there is a straight line that stabs at least $\Omega((\log n/\log\log n)^{1/(d-1)})$ cells. In other words, if a convex subdivision of $d$-space has the property that any line stabs at most $k$ cells, then the subdivision has at most $\exp(O(k^{d-1}\log k))$ cells. This bound is best possible apart from a constant factor. It was previously known only in the case $d = 2$.

**Keywords.** Stabbing number, convex subdivision, space partition, extremal bound.

**AMS Classification.** 05B45, 05D99, 52C22,

## 1 Introduction

A convex subdivision $S$ in $\mathbb{R}^d$ is a set of interior disjoint $d$-dimensional convex sets whose union is $\mathbb{R}^d$. An affine subspace $F$ *stabs* a full-dimensional convex set $\sigma$, if $F$ intersects the interior of $\sigma$. Following earlier results of Chazelle, Edelsbrunner, and Guibas [7], we address combinatorial problems about the stabbing number of subdivisions of the space. It was shown in [7] that for every subdivision of the plane into $n$ convex cells, there is a line that stabs $\Omega(\log n/\log\log n)$ cells[1], and this bound is best possible, which means that the plane can be subdivided into $n$ convex cells such that every line stabs $O(\log n/\log\log n)$ cells. No nontrivial bound has been known for higher dimensional convex subdivisions.

**Contribution.** Denote by $s_d(S)$ the *stabbing number* of a convex subdivision $S$ in $\mathbb{R}^d$, which is the maximum number of convex cells stabbed by a line. For a positive integer $n \in \mathbb{N}$, let

$$s_d(n) = \min\{s_d(S) : S \text{ is a convex subdivision of } \mathbb{R}^d \text{ and } |S| = n\}.$$

**Theorem 1** *For every $d \in \mathbb{N}$, we have*

$$s_d(n) = \Theta\left(\left(\frac{\log n}{\log\log n}\right)^{\frac{1}{d-1}}\right).$$

In other words, if $d$ is fixed and every line stabs at most $k$ cells of a $d$-dimensional convex subdivision $S$, then the size of the subdivision is bounded by $|S| = \exp(O(k^{d-1}\log k))$, and this bound is best possible. The lower bound $s_d(n) = \Omega\left((\log n/\log\log n)^{1/(d-1)}\right)$ is derived in Section 2. The upper bound $s_d(n) = O\left((\log n/\log\log n)^{1/(d-1)}\right)$ is presented in Section 3.

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[1] All logarithms in this paper are of base two, and the exponential function is used as $\exp(x) = 2^x$.

**Related work.** Chazelle and Welzl [8, 16] proved that for every fixed $d \in \mathbb{N}$, any $n$ points in $\mathbb{R}^d$ can be connected by a straight line spanning tree such that any hyperplane is crossed by $O(n^{1-\frac{1}{d}})$ edges of the tree, and this bound is best possible. Matoušek [12] showed that for any $r$, $1 \le r^d \le n$, a set of $n$ points in $\mathbb{R}^d$ can be partitioned into $r^d$ subsets, each of size $\Theta(n/r^d)$, such that every hyperplane stabs the convex hull of $O(r^{d-1})$ subsets. If the points are uniformly distributed in a cube in $\mathbb{R}^d$, then a subdivision into $r^d$ congruent cubes gives such a partition of the point set; the Matoušek partition, however, does not typically correspond to a subdivision of the space.

*Subdivisions* with low stabbing numbers have also been considered. Agarwal, Aronov, and Suri [1] proved that $n$ points in $\mathbb{R}^2$ and in $\mathbb{R}^3$ can be triangulated (with Steiner points) such that every line stabs $O(\sqrt{n}\cdot\log n)$ simplices. They have found point sets in general position such that the stabbing number of any triangulation is $\Omega(\sqrt{n})$. The stabbing number of a *Delaunay* triangulation of $n$ points in $\mathbb{R}^d$, however, may be $\Theta(n^{\lceil d/2 \rceil})$ in the worst case [14]. Bose and Devroye [6] showed recently that the expected stabbing number of the Delaunay triangulation of $n$ points chosen uniformly at random from a compact convex set in the plane is $\Theta(\sqrt{n})$ with high probability.

Hershberger and Suri [11] studied stabbing numbers restricted to the interior of a simple polygon. They showed that every simple polygon with $n$ vertices has a triangulation (with Steiner points) of size $O(n)$ such that any segment lying entirely in the polygon stabs $O(\log n)$ triangles. De Berg and van Kreveld [5] proved that every simple rectilinear polygon with $n$ vertices can be subdivided into $O(n)$ rectangles such that any axis-parallel line segment lying entirely in the polygon stabs $O(\log n)$ rectangles. These results no longer hold if we drop the condition that the stabbing segment lies entirely in the polygon.

Tight bounds are known for the case that both the cells and the stabbing lines are axis-aligned [15]. For every subdivision of $\mathbb{R}^d$, $d \ge 2$, into $n$ axis-aligned boxes, there is an axis-parallel line that stabs $\Omega((\log n)^{1/(d-1)})$ boxes; and this bound is best possible apart from a constant factor.

Instead of the stabbing number of triangulations of a point set in $\mathbb{R}^d$, one can consider the stabbing number of hypergraphs whose edges are non-crossing simplices. Pach [13] asked what is the smallest possible $k$-stabbing number of $e$ distinct $r$-simplices with disjoint relative interiors and having $n$ distinct vertices in $\mathbb{R}^d$, for $1 \le r < k \le d$. Dey and Pach [9] obtained some initial results.

Fekete, Lübbecke, and Meijer [10] have recently proved that it is NP-complete to compute the minimum (axis-parallel) stabbing number of perfect matchings or triangulations for a given point set in the plane. Aronov and Fortune [4] and Aronov *et al.* [3] considered the average stabbing number of triangulations of polygonal scenes.

## 2 Lower bound

In the plane, Chazelle *et al.* [7] argued that if every non-vertical line stabs $O(\log n/\log\log n)$ cells of a convex subdivision of size $n$, then a vertical lines stabs $\Omega(\log n/\log\log n)$ cells: They generated a nested sequence of $\Omega(\log n/\log\log n)$ $y$-monotone polygons where a vertical line intersects a distinct cell clipped to each polygon. This proved a lower bound of $s_2(n) = \Omega(\log n/\log\log n)$. We generalize this argument to arbitrary dimensions $d \ge 2$, replacing $y$-monotone polygons with collections of "vertically convex" polyhedra sandwiched between two cells of the subdivision.

We use the convention that a hyperplane in $\mathbb{R}^d$ parallel to the first $d-1$ coordinate axes ($x_1$, $x_2, \ldots, x_{d-1}$) is *horizontal*, and a line parallel to the $x_d$-axis is *vertical*. This defines the *above-below* relationship in $\mathbb{R}^d$. A set $A \subseteq \mathbb{R}^d$ is *vertically monotone* if its intersection with every vertical line in connected (note, however, that the set $A$ does not have to be connected). For two vertically connected sets $A, B \subset \mathbb{R}^d$, let a *full trapezoid* $C(A, B)$ be the union of *all* vertical segments whose lower endpoint is in the boundary of $A$, upper endpoint in the boundary of $B$, but which is disjoint from the interior of both $A$ and $B$. The union of *some* of these vertical segments that form $C(A, B)$ is called a *partial trapezoid* $C \subseteq C(A, B)$. Clearly, a full or partial vertical trapezoid between any two vertically monotone sets is vertically monotone.

The lower bound in Theorem 1, $s_d(n) = \Omega(\log n / \log\log n)^{1/(d-1)}$ for $d \geq 2$, immediately follows from Lemma 2.

**Lemma 2** *For every $d, k \in \mathbb{N}$, the stabbing number of any convex subdivision of $\mathbb{R}^d$ into at least $\exp(4^{d-1}k^{d-1}\log k)$ convex cells is at least $k$.*

**Proof.** We proceed by induction on $d$. In the base case, $d = 1$, the stabbing number of a subdivision of $\mathbb{R}$ into $n$ convex cells is $n$.

Let $d \in \mathbb{N}$, $d \geq 2$, be an integer and assume that Lemma 2 holds for $d-1$. Consider an integer $k \in \mathbb{N}$ and a convex subdivision $S$ of $\mathbb{R}^d$ into at least $\exp(4^{d-1}k^{d-1}\log k)$ convex cells. We show that there is a vertical line $L$ that stabs at least $k$ cells of $S$ or there is a hyperplane $H$ that stabs at least $\exp(4^{d-2}k^{d-2}\log k)$ cells.

If a hyperplane stabs at least $\exp(4^{d-2}k^{d-2}\log k)$ cells of $S$, then by induction there is a line lying in that hyperplane that stabs $k$ cells of $S$. Assume now that $k \geq 2$ and every hyperplane stabs less than $\exp(4^{d-2}k^{d-2}\log k)$ cells. Let $V$ be the set of all vertices of all $n$ cells of $S$; and let $H^-$ and $H^+$ be horizontal hyperplanes strictly below and, respectively, above the point set $V$. Next we find a vertical line $L$ stabbing $k$ cells.

Proceed in $k$ steps. Initially, let $C_0 = C(H^-, H^+)$ be the horizontal slab between $H^-$ and $H^+$, let $A_0 = H^-$ and $B_0 = H^+$, and let $S_0 = S$. We successively construct partial trapezoids $C_i \subseteq C(A_i, B_i)$, for $i = 1, 2, \ldots, k$, where $A_i, B_i \in S \cup \{H^-, H^+\}$. We denote by $S_i$, $S_i \subseteq S$, the set of convex cells in $S$ whose interiors intersect $C_i$. For $i = 0, 1, \ldots, k-1$, assume that $C_i$ has already been defined with some $A_i, B_i \in S \cup \{H^-, H^+\}$. Choose a hyperplane $H_i$ separating the disjoint convex sets $A_i$ and $B_i$. Denote by $T_i \subseteq S_i$ the set of cells in $S_i$ which are either stabbed by $H_i$ or have a $(d-1)$-dimensional face in $H_i$ and lie in the closed halfspace above $H_i$. There is a hyperplane (obtained by slightly perturbing $H_i$) that stabs all cells in $T_i$ , and so there are less than $\exp(4^{d-2}k^{d-2}\log k)$ cells in $T_i$. If a point in $C_i$ is not in a cell of $T_i$, then it must be vertically above some cell in $T_i \cup \{A_i\}$ and vertically below some cell in $T_i \cup \{B_i\}$. Therefore, $C_i \setminus \bigcup T_i$ is partitioned into up to $\binom{|T_i|+2}{2} - 1$ partial trapezoids between two convex sets in $T_i \cup \{A_i, B_i\}$. We have $\binom{|T_i|+2}{2} - 1 \leq 2|T_i|^2$. Let $C_{i+1}$ be the one of these partial trapezoids that intersects the largest number of cells of $S$. The partial trapezoid $C_{i+1} \subseteq C(A_{i+1}, B_{i+1})$ is sandwiched between some $A_{i+1} \in T_i \cup \{A_i\}$ and $B_{i+1} \in T_i \cup \{B_i\}$. The number of cells intersecting $C_{i+1}$ is

$$|S_{i+1}| \geq \frac{|S_i| - |T_i|}{2|T_i|^2}.$$

We show that $|S_i| \geq \exp(4^{d-1}(k-i)k^{d-2}\log k) \geq 1$ for $i = 0, 1, \ldots, k$. Initially for $i = 0$, we have $|S_0| = |S| \geq \exp(4^{d-1}k^{d-1}\log k)$. If $|S_i| \geq \exp(4^{d-1}(k-i)k^{d-2}\log k)$ for some $i = 0, 1, 2, \ldots, k-1$, then we have

$$
\begin{aligned}
|S_{i+1}| &\geq \frac{|S_i|-|T_i|}{2|T_i|^2} \geq \frac{\exp(4^{d-1}(k-i)k^{d-2}\log k)-\exp(4^{d-2}k^{d-2}\log k)}{\exp(1+2\cdot 4^{d-2}k^{d-2}\log k)} \\
&\geq \frac{\exp(4^{d-1}(k-i)k^{d-2}\log k)}{\exp(2+2\cdot 4^{d-2}k^{d-2}\log k)} \geq \frac{\exp(4^{d-1}(k-i)k^{d-2}\log k)}{\exp(4^{d-1}k^{d-2}\log k)} \\
&\geq \exp(4^{d-1}(k-i-1)k^{d-2}\log k).
\end{aligned}
$$

Note that a vertical line $L$ stabbing $C_k$ must meet one convex cell in $T_i$ for each $i = 0, 1, \dots, k-1$. The collections of cells $T_i$ are disjoint. Hence, $L$ stabs at least $k$ cells of $S$, one from each $T_i$, $i = 0, 1, \dots, k-1$. □

## 3 Upper bound construction

A subdivision of a $d$-dimensional unit cube $B_d$ can be extended into the the subdivision of any convex bounding set $D$, with $B_d \subseteq D \subseteq \mathbb{R}^d$, by adding $2d$ convex cells, each of which is the intersection of the exterior of $B_d$ and a cone with apex at a point in the interior of $B_d$ and spanned by a facet of $B_d$. These $2d$ additional "filler" convex cells increase the stabbing number by an additive term of no more than $2d$.

Note that the stabbing number is an affine invariant, and so $B_d$ and its subdivision can be deformed by an affine transformation without changing the stabbing number. We represent the direction of a line $L$ in $\mathbb{R}^d$ by two antipodal points in $\mathbb{S}^{d-1}$, which span a line parallel to $L$ and incident to the origin. The antipodal point pairs in $\mathbb{S}^{d-1}$ are endowed with the usual angle metric, where the diameter of $\mathbb{S}^{d-1}$ is 90°. The *δ-neighborhood* of an antipodal set $A \subseteq \mathbb{S}^{d-1}$ is the set $\{p \in \mathbb{S}^{d-1} : \exists q \in A, \mathrm{dist}(p,q) \leq \delta^\circ\}$. The line directions orthogonal to a given line $L$ lie on a *hyper-sphere* of $\mathbb{S}^{d-1}$, which is a subset congruent to $\mathbb{S}^{d-2}$.

The upper bound in Theorem 1, $s_d(n) = O((\log n/\log\log n)^{1/(d-1)})$ for $d \geq 2$, immediately follows from Lemma 3.

**Lemma 3** *For every $d, k \in \mathbb{N}$, $k \geq (12d^2)^{d-1}$, there is a convex subdivision of $B_d$ into at least $\exp(c_d k^{d-1}\log k)$ convex cells with $c_d = 6^{1-d}\Pi_{i=1}^d (9i^2)^{1-i}$ such that any line stabs at most $k$ cells.*

**Proof.** For every $d, k \in \mathbb{N}$, $k \geq (12d^2)^{d-1}$, we construct a convex subdivision $S_d(k)$ of a $d$-dimensional unit cube $B_d$ into at least $\exp(c_d k^{d-1}\log k)$ convex cells with stabbing number at most $k$. We guarantee the additional property that every point lying on an $f$-dimensional face of $B_d$, for $f = 0, 1, \dots, d$, is incident to at most $f+1$ convex cells of $S_d(k)$.

Proceed by induction on $d$. For $d = 1$, let $S_1(k)$ be the partition of the unit interval $B_1$ into $k$ congruent intervals. Clearly, the stabbing number of $S_1(k)$ is $k$, each endpoint of $B_1$ is incident to one cell, and every point in the interior of $B_1$ is incident to at most two cells.

Let $d \geq 2$, and assume that for every $k \in \mathbb{N}$, $k \geq (12(d-1)^2)^{d-2}$, there is a convex subdivision $S_{d-1}(k)$ of the $(d-1)$-dimensional unit cube $B_{d-1}$ into at least $\exp(c_{d-1}k^{d-2}\log k)$ convex cells, with stabbing number at most $k$, and such that every point lying on an $f$-dimensional face of $B_{d-1}$ is incident to at most $f+1$ cells. We next construct a subdivision $S_d(k)$ of a $d$-dimensional unit cube $B_d$ for every $k \in \mathbb{N}$, $k \geq (12d^2)^{d-1}$.

We construct $S_d(k)$ in $\ell = \lfloor k/(6d^2) \rfloor$ steps. Specifically, we construct a sequence of subdivisions $\Sigma_i$, for $i = 0, 1, \ldots, \ell$, of $B_d$. Initially, let $\Sigma_0 = \{B_d\}$, and the last element of the sequence will be $\Sigma_\ell = S_d(k)$. For each subdivision $\Sigma_i$, we distinguish *seed* and *filler* cells. Initially, let $B_d$ be a seed cell of $\Sigma_0$. We construct $\Sigma_i$, for $i = 1, 2, \ldots, \ell$, by subdividing each seed cell of $\Sigma_{i-1}$ into subcells. We next construct an auxiliary subdivision: Let $X(\delta)$ be a $d$-dimensional subdivision of the axis-aligned box $B_{d-1} \times [0, \delta]$ obtained by taking the cross product of the cells of $S_{d-1}(\ell)$ with an orthogonal interval $[0, \delta]$, where $\delta > 0$ is a parameter to determined in each step $i$ independently. Choose a set $\mathcal{H}$ of $\ell$ congruent copies of $\mathbb{S}^{d-2}$ in $\mathbb{S}^{d-1}$ in general position, that is, such that every point in $\mathbb{S}^{d-1}$ is contained in at most $d-1$ hyper-spheres of $\mathcal{H}$. Label the hyper-spheres of $\mathcal{H}$ as $H_1, H_2, \ldots, H_\ell$ arbitrarily. We use the convention that for $0 \leq f < d$, the $f$-dimensional sphere $\mathbb{S}^f$ is embedded into $\mathbb{R}^d$ as a unit sphere in the $(f+1)$-flat spanned by the first $f+1$ coordinate axes.

Given a subdivision $\Sigma_{i-1}$, we construct $\Sigma_i$ as follows. Choose a set $P_i$ of points, one point in the interior of each seed cell of $\Sigma_{i-1}$, such that there are no three collinear points, and the direction of the line spanned by any two points is not in any hyper-sphere in $\mathcal{H}$. Let $\varepsilon_i > 0$ be a sufficiently small constant such that the direction of the line spanned by any two points in $P_i$ is not in the $2\varepsilon_i$-neighborhood of any hyper-sphere in $\mathcal{H}$. Let $\varrho_i > 0$ be a sufficiently small constant such that each ball of radius $\varrho_i$ centered at a point in $P_i$ lies in the interior of a seed cell of $\Sigma_{i-1}$; no line stabs three such balls; and the direction of a line that stabs two balls is not in the $\varepsilon_i$-neighborhood of a hyper-sphere in $\mathcal{H}$.

Now, let $\delta_i > 0$ be a sufficiently small constant such that the direction of any line that intersects $d$ cells of $X(\delta_i)$ lies in the $\varepsilon_i$-neighborhood of $\mathbb{S}^{d-2}$. Note that a vertex of a cell in $S_{d-1}(\ell)$ may be incident to up to $d$ convex cells, and so a line in the proximity of the edge of $X(\delta_i)$ parallel to the $x_d$-axis may already stab $d$ cells of $X(\delta_i)$. Transform $X(\delta_i)$ with a congruency that maps the $x_d$-axis to a line orthogonal to $H_i$; and scale it down such that it fits into the interior of a ball of radius $\varrho_i$. For constructing $\Sigma_i$, place such an affine copy of $X(\delta_i)$ into the interior of the ball of radius $\varrho_i$ centered at each point of $P_i$. Fill the remaining space in each seed cell $\sigma \in \Sigma_{i-1}$ by $2d$ convex cells, each of which is the intersection of $\sigma$ and a cone whose apex is in the interior of the affine copy of $X(\delta_i)$ and is spanned by a facet of the affine copy of the box $B_{d-1} \times [0, \delta_i]$. All $2d$ cones have the same apex, which can be chosen such that an $(f+1)$-flat spanned by the apex and an $f$-dimensional face of the affine copy of $X(\delta_i)$ does not contain any face of $\sigma$. This guarantees that in the subdivision of a seed cell $\sigma$, every point on an $f$-dimensional face is incident to at most $f+1$ convex cells, for $f = 0, 1, \ldots, d$. The resulting subdivision is $\Sigma_i$. Let the cells in the affine copies of $X(\delta_i)$ be the seed cells of $\Sigma_i$, and let the remaining cells be filler cells.

$\Sigma_0$ consists of a single (seed) cell $B_d$ and each of the $\ell$ steps increases the number of seed cells by a factor of $|S_{d-1}(\ell)| = \exp(c_{d-1}\ell^{d-2}\log \ell)$. In the resulting subdivision $\Sigma_\ell = S_d(k)$, the number of seed cells (hence the total number of cells) is at least

$$\begin{aligned}
|S_d(k)| &\geq |S_{d-1}(\ell)|^\ell \geq (\exp(c_{d-1}\ell^{d-2}\log \ell))^\ell = \exp(c_{d-1}\ell^{d-1}\log \ell) \\
&= \exp\left(c_{d-1}\left\lfloor \frac{k}{6d^2} \right\rfloor^{d-1} \log \left\lfloor \frac{k}{6d^2} \right\rfloor\right) > \exp\left(c_{d-1}\left(\frac{k}{9d^2}\right)^{d-1} \frac{\log k}{6}\right) \\
&= \exp\left(\frac{c_{d-1}}{6(9d^2)^{d-1}} \cdot k^{d-1}\log k\right) = \exp(c_d k^{d-1}\log k).
\end{aligned}$$

We show next that the stabbing number of $\Sigma_\ell$ is at most $k$. Let $L$ be a line in $\mathbb{R}^d$. Consider the seed cells created in all $\ell$ levels of the recursion. For $i = 1, 2, \ldots, \ell$, each seed cell created at level

$i$ is nested in a seed cell at level $i-1$. For $i = 0, 1, \dots, \ell - 1$, each seed cell contains at most $2d$ filler sells that are interior disjoint from seed cells on lower levels of the recursion. So it is enough to bound the number of seed cells stabbed by $L$ and created at various levels of the recursion. If the direction of $L$ lies in the $\varepsilon_i$-neighborhood of $H_i$, then $L$ stabs at most $\ell$ seed cells at level $i$, all in the same affine copy of $X(\delta_i)$. If the direction of $L$ does not lie in the $\varepsilon_i$-neighborhood of $H_i$, then $L$ stabs at most two affine copies of $X(\delta_i)$ and stabs at most $d$ seed cells in each copy, that is, $L$ stabs at most $2d$ seed cells at level $i$. The direction of a line $L$ lies in the $\varepsilon_i$-neighborhood of $H_i$ for at most $d-1$ hyper-spheres $H_i \in \mathcal{H}$. Hence $L$ stabs at most $(d-1)\ell + \ell 2d < 3d\ell$ seed cells and at most $(3d\ell)(2d) = 6d^2\ell \leq k$ convex cells of $\Sigma_\ell$. □

## 4 Further directions

One can define the *k-stabbing number*, $s_{d,k}(S)$, of a convex subdivision $S$ in $\mathbb{R}^d$ for any $0 \leq k \leq d$ as the maximum number of convex cells stabbed by a $k$-flat. For a positive integer $n \in \mathbb{N}$, let

$$s_{d,k}(n) = \min\{s_{d,k}(S) : S \text{ is a convex subdivision of } \mathbb{R}^d \text{ and } |S| = n\}.$$

We have $s_{d,0}(n) = 1$, $s_{d,1}(n) = s_d(n) = \Theta((\log n / \log\log n)^{1/(d-1)})$, and $s_{d,d}(n) = n$. It remains an open problem to determine the asymptotic behavior of $s_{d,k}(n)$ for all $d, k \in \mathbb{N}$ with $1 < k < d$.

It is not clear whether Theorem 1 continues to hold if we impose restrictions on the convex subdivision $S$. A polyhedral subdivision $S$ in $\mathbb{R}^d$ is called *face-to-face* if every face of a cell in $S$ is a face of all adjacent cells. A full-dimensional compact convex polytope in $\mathbb{R}^d$ is called *$\alpha$-fat*, for a constant $\alpha \geq 1$, if the ratio of radii of the smallest enclosing ball and the largest inscribed ball is bounded by $\alpha$. Our upper bound construction is not a face-to-face subdivision, and the compact cells are not $\alpha$-fat for any fixed $\alpha$ (while $n$ tends to infinity). What is the minimum stabbing number of a face-to-face convex subdivision of size $n$ in $\mathbb{R}^d$, $d \geq 2$? What is the minimum stabbing number of a convex subdivision of size $n$ of a cube in $\mathbb{R}^d$ where all cells are $\alpha$-fat for constants $\alpha \geq 1$ and $d \geq 2$?

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