Upper bound constructions for untangling planar geometric graphs*

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Abstract

For every $n \in \mathbb{N}$, we construct an n-vertex planar graph $G = (V, E)$, and n distinct points $p(v)$, $v \in V$, in the plane such that in any crossing-free straight-line drawing of G, at most $O(n^{.4948})$ vertices $v \in V$ are embedded at points $p(v)$. This improves on an earlier bound of $O(\sqrt{n})$ by Goaoc *et al.* [10].

1 Introduction

A graph $G = (V, E)$ is defined by a vertex set V and an edge set $E \subseteq \binom{V}{2}$. A *straight-line drawing* of a graph $G = (V, E)$ is a mapping $p : V \to \mathbb{R}^2$ of the vertices into distinct points in the plane, which induces a mapping of the edges $\{u, v\} \in E$ to line segments $p(u)p(v)$ between the corresponding points. A straight-line drawing is *crossing-free* if no two edges intersect, except perhaps at a common endpoint. By Fáry's theorem [9], a graph is planar if and only if it admits a crossing-free straight-line drawing.

Suppose we are given a planar graph $G = (V, E)$ and a straight-line drawing $p : V \to \mathbb{R}^2$ (in which some edges may cross each other). Since G is planar, we can obtain a *crossing-free* straight-line drawing $p' : V \to \mathbb{R}^2$ by moving some of the vertices to new positions in the plane. The process of changing a drawing $p : V \to \mathbb{R}^2$ to a *crossing-free* drawing $p' : V \to \mathbb{R}^2$ is called an *untangling* of (G, p). A vertex $v \in V$ is *fixed* in the untangling if $p(v) = p'(v)$. Denote by $f(n)$, $n \in \mathbb{N}$, the maximum integer such that every straight-line drawing of every n-vertex planar graph can be untangled while keeping at least $f(n)$ vertices fixed. In this paper, we study the asymptotic growth rate of $f(n)$.

The first question on untangling planar graphs was posed by Mamoru Watanabe in 1998: Is it true that every polygon P with n vertices can be untangled in at most ϵn steps, for some absolute constant $\epsilon < 1$, where in each step, we move a vertex of P to a new location? Pach and Tardos [16] gave a negative answer to Watanabe's question. They showed that every polygon with n vertices (i.e., the straight-line drawing of the cycle C_n) can be untangled in at most $n - \sqrt{n}$ moves, and there are n-vertex polygons where no more than $O((n \log n)^{2/3})$ vertices can be fixed. Recently, Cibulka [6] proved that every n-vertex polygon can be untangled such that $\Omega(n^{2/3})$ vertices are fixed.

The problem of untangling planar graphs was studied by Goaoc *et al.* [10]. They proved $f(n) \leq \sqrt{n} + 2$ by constructing drawings of the planar graph $P_2 * P_{n-2}$ with n vertices such that any

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untangling fixes at most $\sqrt{n}+2$ vertices. Here P_k denotes a path with k vertices; and for two graphs, G and H, the join $G * H$ consists of the vertex-disjoint union of G and H and all edges between $V(G)$ and $V(H)$, see Fig. 1. Bose *et al.* [3], Kang *et al.* [15] and Ravsky and Verbitsky [17] explored drawings of several families of n-vertex planar graphs in which any untangling fixes $O(\sqrt{n})$ vertices. Cibulka [6] showed that every 3-connected planar graph with n vertices admits a drawing for which any untangling fixes $O((n \log n)^{2/3})$ vertices.

Bose *et al.* [3] devised an algorithm that untangles every geometric graph with n vertices while fixing $(n/3)^{1/4}$ vertices, which proves $f(n) \geq (n/3)^{1/4}$.

In this paper, we improve the upper bound for $f(n)$ from $O(\sqrt{n})$ to $O(n^{.4948})$. We construct an n-vertex planar graph $G = (V, E)$ and arrange the vertices in the plane by an injective mapping $p : V \to \mathbb{R}^2$ such that any untangling of (G, p) fixes $O(n^{2/(7-3\lambda)+\varepsilon})$ vertices for every $\varepsilon > 0$, where λ is the shortness exponent of the family of 3-connected cubic planar graphs (i.e., polyhedral graphs). The exact value of the shortness exponent λ is not known. The currently known best upper bound is $\lambda \leq \log_{23} 22 \approx 0.9858$ by Grünbaum and Walther [11]. Any improvement on the upper bound for λ would immediately improve our upper bound for $f(n)$.

Our argument crucially depends on a correspondence between the shortness exponent of cubic polyhedral graphs and crossing-free straight-line drawings of the dual graph in which a line stabs many faces. We define the *stabbing number* $\text{stab}(G)$ of a polyhedral graph $G = (V, E)$ to be the maximum number of faces that intersect a line L in any crossing-free straight-line drawing of G. We prove that $\text{stab}(G)$ is the size of the maximum cycle in the dual graph G^*. Previously, Biedl *et al.* [1] proved a similar but weaker result for polyline drawings (rather than straight-line drawings).

Organization. In Section 2, we discuss two key ingredients of our construction: (i) the shortness exponent of cubic polyhedral graphs, and (ii) permutations with certain special properties related to the Erdős-Szekeres Theorem. In Section 3, we present a family of planar geometric graphs and prove $f(n) \in O(n^{2/(7-3\log_{23} 22)+\varepsilon})$. We conclude in Section 4 by establishing a correspondence between the shortness exponent of cubic polyhedral graphs and the stabbing number of triangulations.

2 Preliminaries

Dual graphs of triangulations. The value of $f(n)$ is attained for edge-maximal planar graphs with n vertices, since by augmenting a planar graph with new edges, the set of its crossing-free straight-line drawings decreases or remains the same. The edge-maximal planar graphs are called *triangulations*. By Euler's formula, a triangulation with $n \geq 3$ vertices has exactly $3n-6$ edges and $2n-4$ faces (including the outer face). Note that in every crossing-free drawing of a triangulation, every face (including the outer face) is bounded by three edges. It follows that every triangulation with $n \geq 4$ vertices is 3-connected [7][Lemma 4.4.5].

The 3-connected planar graphs are also called *polyhedral graphs*. By Whiteley's theorem, every polyhedral graph has a topologically unique crossing-free drawing, apart from the choice of the outer face. More precisely, in every crossing-free drawing of a 3-connected graph G, the face boundaries are precisely the nonseparating chordless cycles of G [7][Proposition 4.2.7]. Hence, G has a well-defined dual graph G^* (independent of the crossing-free drawings of G): the vertices of G^* correspond to the faces (i.e., nonseparating chordless cycles) of G, and two vertices of G^* are adjacent if and only if the corresponding faces share an edge. If G is a triangulation with $n \geq 4$ vertices, then G^* is a cubic polyhedral graph with $2n-4$ nodes and $3n-6$ edges.

Stabbing triangulations and dual cycles. The following observation is crucial for our construction.

Observation 1 *Let T be a polyhedral graph. Suppose that a line L stabs the faces $f_1, \ldots, f_k$ (in this order) in a crossing-free straight-line drawing of T, and these faces correspond to the vertices $f_1^*, \ldots, f_k^*$, respectively, in the dual graph T^*. Then $(f_1^*, \ldots, f_k^*)$ is a simple cycle in T^*.*

In Section 3, we will construct a planar graph G from two triangulations, S and T. Specifically, we plug a copy of S in each face of T. We then draw G in the plane such that the vertices of every copy of S lie on a line L. If the dual graph T^* is not Hamiltonian, then in any crossing-free straight-line drawing of G, the line L will miss at least one face of T. If L misses a face f of T, then none of the vertices can be fixed in the copy of S plugged into f. In the next few paragraphs, we review the currently known best bounds on the maximum cycles in the dual graphs of triangulations.

In Section 4, we establish a somewhat surprising converse of Observation 1, and show that if $(f_1^*, \ldots, f_k^*)$ is a simple cycle in the dual graph T^* of a polyhedral graph T, then T admits a crossing-free straight-line drawing in which a line L stabs the corresponding faces $f_1, \ldots, f_k$ of T in this order.

Maximum cycles in cubic polyhedral graphs. In an attempt at proving the Four Color Theorem, Tait [18] conjectured in 1884 that every cubic polyhedral graph is Hamiltonian. In 1946, Tutte [20] found a counterexample with 44 vertices. The smallest known counterexample, due to Barnette, Bosák, and Lederberg, has 38 vertices, and it is known that there is no counterexample with 36 or fewer vertices [12]. Using the smallest known counterexample to Tait's conjecture, one can build a cubic polyhedral graph with $\Theta(n)$ vertices for every $n \in \mathbb{N}$ in which every cycle has at most $O(n^{\log_{37} 36}) \subset O(n^{0.9925})$ vertices. Using similar techniques, Grünbaum and Walther [11] constructed for every $n \in \mathbb{N}$ a cubic polyhedral graph with $\Theta(n)$ vertices in which every cycle has at most $O(n^{\log_{23} 22}) \subset O(n^{0.9859})$ vertices.

Shortness exponent. The *shortness exponent* of a family of graphs was introduced by Grünbaum and Walther [11]. For a graph G, let $V(G)$ denote the set of vertices of G and let $h(G)$ be the number of vertices in a longest cycle in G (also known as the *circumference* of G). The shortness exponent of an infinite family $\mathcal{G}$ of graphs is

$$\lambda(\mathcal{G}) = \liminf_{n\to\infty} \frac{\log h(G_n)}{\log |V(G_n)|},$$

where $(G_n)_{n=1}^{\infty}$ is the sequence of all graphs in $\mathcal{G}$. This means that for every $\varepsilon > 0$, there are arbitrarily large graphs $G \in \mathcal{G}$ that contain a cycle of length $|V(G)|^{\lambda(\mathcal{G})-\varepsilon}$.

For example, the shortness exponent is 1 for the family of Hamiltonian graphs, and 0 for the family of forests. The shortness exponent of cubic polyhedral graphs is not known. The currently known best lower bound, due to Bilinski *et al.* [2], is $\lambda \geq x \approx 0.7532$, where x is the real root of $4^{1/x} - 3^{1/x} = 2$. The currently known best upper bound is $\lambda \leq \log_{23} 22 \approx 0.9858$ due to Grünbaum and Walther [11].

Monotone subsequences. Erdős and Szekeres [8] showed that every permutation of $\{1, \ldots, n\}$ contains a monotonically increasing or decreasing subsequence of length at least $\lceil \sqrt{n} \rceil$, and this bound is the best possible. The lower bound is attained on many different permutations. Perhaps the simplest construction consists of $\lceil\sqrt{n}\rceil$ monotonically increasing subsequences of consecutive elements, where the minimum element of each subsequence is larger than the maximum element of the next. We will use a permutation where the monotone sequences are "spread out" more evenly. In a permutation $(\sigma_1, \sigma_2, \ldots, \sigma_n)$, we define the *spread* of a subsequence $(\sigma_{j_1}, \sigma_{j_2}, \ldots, \sigma_{j_k})$, $1 \le j_1 < j_2 < \ldots. < j_k \le n$, to be $j_k - j_1$. In other words, the spread of a subsequence is the length of the minimum interval in $[1, n]$ that contains all indices of the subsequence.

Lemma 1 *For every $m \in \mathbb{N}$, let $n = 4^m$. Then there is a permutation π_n of $\{1, \ldots, n\}$ such that*

- *the length of every monotone subsequence is at most $2^m = \sqrt{n}$, and*
- *for every $k \ge 2$, the spread of every monotone subsequence of length k is at least $\frac{k^2+2}{6}$.*

Proof. We construct the permutation π_n by induction on m. For $m = 1$, let $\pi_4 = (3, 4, 1, 2)$ and observe that it has the desired properties. Assume that $\pi_n = (\sigma_1, \ldots, \sigma_n)$ is a permutation of $\{1, \ldots, n\}$ with the desired properties. We construct a permutation π_{4n} of $\{1, \ldots, 4n\}$ by replacing each σ_i with the 4-tuple

$$(4(\sigma_i - 1) + 3, 4(\sigma_i - 1) + 4, 4(\sigma_i - 1) + 1, 4(\sigma_i - 1) + 2).$$

Let τ be a monotone subsequence of length k in π_{4n}. Note that τ has at most two elements from each 4-tuple. The sequence of these 4-tuples corresponds to a monotone subsequence of π_n, which we denote by τ'. By the pigeonhole principle, the length of τ' is at least $k/2$, with equality iff τ contains exactly two elements from each of the 4-tuples involved. By induction, the length of τ' is at most 2^m. Hence, we have $k \le 2^{m+1}$, as required.

For the proof of the second claim, we distinguish between three cases.

1. *The length of τ' is exactly $k/2$.* If $k = 2$, then the spread of τ is clearly at least $\frac{2^2+2}{2} = 1$. If $k/2 \ge 2$, then the spread of τ' is at least $\frac{(k/2)^2+2}{6}$ in π_n by induction. As noted above, τ contains exactly two elements from each of the 4-tuples involved. Consequently, the spread of τ is minimized when τ contains the last two elements of the first 4-tuple, and the first two elements of the last 4-tuple. In this case, the spread of τ is at least $4 \cdot \frac{(k/2)^2+2}{6} - 1 = \frac{k^2+2}{6}$.

2. *The length of τ' is exactly $(k+1)/2$.* In this case, k is odd and $k \ge 3$. Since $(k+1)/2 \ge 2$, the spread of τ' is at least $\frac{(k+1)^2/4+2}{6}$ by induction, and contains exactly two elements from all but one of the 4-tuples involved. Consequently, the spread of τ is at least $4 \cdot \frac{(k+1)^2/4+2}{6} - 2 = \frac{k^2+2k-3}{6} > \frac{k^2+2}{6}$.

3. *The length of τ' is more than $\lceil k/2 \rceil$.* By induction, the spread of τ' is at least $\frac{(k/2+1)^2+2}{6}$. The spread of τ is minimized when τ contains the last element of the first 4-tuple, and the first element of the last 4-tuple. In this case, the spread of τ is at least $4 \cdot \frac{(k/2+1)^2+2}{6} - 3 = \frac{k^2+4k-6}{6} \ge \frac{k^2+2}{6}$.

In all three cases, the spread of τ is at least $\frac{k^2+2}{6}$, as claimed. $\square$

3 Upper Bound Construction

Theorem 1 *For every $\varepsilon > 0$, we have $f(n) \in O(n^{2/(7-3\lambda)+\varepsilon})$, where λ is the shortness exponent of the family of cubic polyhedral graphs, and the constant of proportionality depends on ε.*

Proof. Fix $\varepsilon > 0$. Let $\delta \in (0, \frac{1}{100})$ be a sufficiently small constant such that

$$0 < \frac{2}{7-3(\lambda+\delta)} \leq \frac{2}{7-3\lambda} + \varepsilon.$$

For every $n_0 \in \mathbb{N}$, we construct a planar graph $G = (V, E)$ with n vertices and a straight-line drawing $p : V \to \mathbb{R}^2$ such that $n \geq n_0$ and every untangling of G fixes $O(n^{2/(7-3\lambda)+\varepsilon})$ vertices.

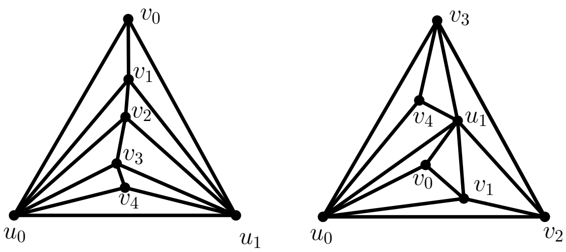

Figure 1: Two crossing-free straight-line drawings of the triangulation $S = P_2 * P_5$, where $P_2 = (u_0, u_1)$ and $P_5 = (v_0, \ldots, v_5)$.

Construction. For every $n_0 \in \mathbb{N}$, we first construct a planar graph $G = (V, E)$ on $n \geq n_0$ vertices, and then describe an embedding $p : V \to \mathbb{R}^2$. Let $\kappa \in (0, 1)$ be a constant to be determined later. By the definition of the shortness exponent, there exists a cubic polyhedral graph T with m vertices such that $m \geq n_0^\kappa$ and every cycle in T has at most $m^{\lambda+\delta}$ vertices. The dual graph of T is a 3-connected triangulation T^* with m faces and $m/2 + 2$ vertices. In every crossing-free straight-line drawing of T^*, every line stabs at most $m^{\lambda+\delta}$ triangular faces by Observation 1.

Let $s \in \mathbb{N}$ be the smallest power of 4 with $s \geq m^{1/\kappa - 1}$. Let S be the join $P_2 * P_{s+1}$ of the paths $P_2 = (u_0, u_1)$ and $P_{s+1} = (v_0, \ldots, v_s)$, see Fig. 1. We construct G by combining the triangulation T^* with m isomorphic copies of S, denoted $S_1, \ldots, S_m$. Specifically, label the m faces of T^* by $f_1^*, \ldots, f_m^*$ arbitrarily, and identify the vertices u_1, u_2, and v_0 of S_i with the three vertices of f_i^* (in an arbitrary order). Note that the vertices $v_1, \ldots, v_s$ of S_i have not been identified with any other vertices, we say that these s vertices of S_i are *interior vertices* for $i = 1, \ldots, m$. With this terminology, G has ms interior vertices, and the total number of vertices of G is $n = (m/2 + 2) + ms > ms \geq m \cdot m^{1/\kappa - 1} = m^{1/\kappa} \geq n_0$, as required.

Next, we describe a straight-line drawing of G. The 3-connected triangulation T^* with $m/2 + 2$ vertices has a straight-line drawing such that in every untangling of T at most $O((m \log m)^{2/3})$ vertices are fixed [6], and all vertices lie strictly above the x-axis. Embed the interior vertices $\{v_1, \ldots, v_s\}$ of S_1 into integer points $\{(1, 0), \ldots, (s, 0)\}$ such that v_j is mapped to $(\pi_s(j), 0)$, for $j = 1, \ldots, s$, where π_s is a permutation described in Lemma 1. For $i = 2, \ldots, m$, embed the interior vertices of S_i into translated copies of these points, each translated along the x-axis by $\frac{i-1}{2m}$. Since $0 < \frac{i-1}{2m} < 1$, all interior vertices of G are mapped to distinct points on the interval $[1, s+1]$ of the x-axis.

Bounding the number of fixed vertices. Consider a crossing-free straight-line drawing of G. This drawing induces a crossing-free drawing of the subgraph T^* of G. By construction, at most

$$O(m^{2/3}\log^{2/3} m) \tag{1}$$

of the $m/2+2$ vertices of T^* are fixed. By Observation 1, the x-axis intersects at most $O(m^{\lambda+\delta})$ faces of T^* in any crossing-free drawing. By Whiteley's theorem, the s interior vertices of S_i must lie in the interior of the face f_i^* of T^*. Denote by ℓ_i the number of fixed interior vertices of S_i, for $i=1,2,\ldots,m$. If a face f_i^* of T^* is disjoint from the x-axis, then none of the interior vertices of S_i is fixed. Therefore, all but at most $O(m^{\lambda+\delta})$ values of ℓ_i are zero.

Assume, by relabeling the copies of S if necessary, that $\ell_i > 0$ for the first $r+1$ values of i, and $\ell_i = 0$ for all other values of i, where $r \in O(m^{\lambda+\delta})$. Since T^* has only one unbounded face in our straight-line drawing, we may also assume that faces f_i^*, $i=1,\ldots,r$, are bounded (and face f_{r+1}^* may be unbounded). Consider a triangulation S_i with $1 \le i \le r$. Then the interior vertices of S_i lie in the interior of the straight-line drawing of the triangle (u_0,u_1,v_0) as in Fig. 1(left). Consequently, the triangle (u_0,u_1,v_j) contains the triangle (u_0,u_1,v_{j+1}) for $j=0,\ldots,s-1$. The intersection of the x-axis with the interior of each these triangles is a line segment. First consider the case that the x-axis does not intersect the common edge u_0u_1 of these nested triangles. Then the x-axis contains at most one of the interior vertices of S_i, and so $\ell_i = 1$. Otherwise, the x-axis intersects the vertices of the ℓ_i nested triangles in the order in which they are nested. Therefore at least $\ell_i \ge 2$ fixed interior points of S_i form a monotone subsequence in the permutation π_s.

By Lemma 1, the spread of a monotone subsequence of length ℓ_i is at least $(\ell_i^2+2)/6$ when $\ell_i \ge 2$, and 0 when $\ell_i = 1$. In both cases, the spread of a monotone subsequence of length ℓ_i is at least $(\ell_i^2-1)/6$. Consequently, the face f_i^* intersects the interval $[1,s+1]$ of the x-axis in an interval of length at least $(\ell_i^2-1)/6$. Distinct faces of T^* intersect the x-axis in disjoint intervals. We conclude that

$$\sum_{i=1}^{r} \frac{\ell_i^2-1}{6} \le s. \tag{2}$$

Recall that $r \in O(m^{\lambda+\delta})$. By the inequality between arithmetic and harmonic means, the sum $\sum_{i=1}^{r}\ell_i$ is maximal when all nonzero values of ℓ_i are equal. In this case, Inequality (2) becomes $r(\ell^2-1) \in O(m^{1/\kappa-1})$, that is, $\ell \in O(m^{(1/\kappa-\lambda-\delta-1)/2})$. Therefore, the total number of fixed interior vertices of S_i, for $i=1,2,\ldots,m$, is

$$\sum_{i=1}^{m}\ell_i = \sum_{i=1}^{r+1}\ell_i \le r\ell + s \in O(m^{(1/\kappa+\lambda+\delta-1)/2} + m^{1/\kappa-1}). \tag{3}$$

We choose the constant κ such that

$$\begin{aligned} m^{(1/\kappa+\lambda+\delta-1)/2} &= m^{2/3} \\ \frac{1/\kappa+\lambda+\delta-1}{2} &= \frac{2}{3} \\ \kappa &= \frac{3}{7-3(\lambda+\delta)}. \end{aligned}$$

Since $0.7432 \le \lambda \le 0.9857$ and $0 < \delta < 0.01$, we have $0.62 < \kappa < 0.75$ and $1/\kappa - 1 < 2/3$. Substituting this value of κ into (3), we obtain

$$\sum_{i=1}^{m}\ell_i \in O(m^{(1/\kappa+\lambda+\delta-1)/2} + m^{1/\kappa-1}) \subseteq O(m^{2/3}). \tag{4}$$

From (1), the number of fixed vertices in the triangulation T^* is at most $O(m^{2/3}\log^{2/3} m) = O(n^{2\kappa/3}\log^{2/3} n) = O(n^{2/(7-3(\lambda+\delta))}\log^{2/3} n) \subset O(n^{2/(7-3\lambda)+\varepsilon})$. From (4), the number of fixed interior vertices is bounded by $O(m^{(1/\kappa+\lambda+\delta-1)/2}) = O(m^{2/3}) \subset O(n^{2/(7-3\lambda)+\varepsilon})$. We conclude that the total number of fixed vertices in any untangling of G is $O(n^{2/(7-3\lambda)+\varepsilon})$, as required. □

Combining Theorem 1 with the upper bound $\lambda \le \log_{23} 22$ by Grünbaum and Walther [11], we obtain the following.

Corollary 1 *For every $\varepsilon > 0$, we have $f(n) \in O(n^{2/(7-3\log_{23}22)-\varepsilon)}) \subset O(n^{.4948})$, where the constant of proportionality depends on ε.*

4 Stabbing number of triangulations

In this section, we prove the converse of Observation 1: If T is a polyhedral graph and $(f_1^*, \ldots, f_k^*)$ is a simple cycle in the dual graph T^*, then T admits a crossing-free straight-line drawing where some line stabs the faces $f_1, \ldots, f_k$ in this order. We construct the straight-line drawing of T incrementally, based on Lemmas 2 and 3 below.

Recall that a *near-triangulation* is a 2-connected planar graph such that all faces are triangles with the possible exception of one face, which is considered to be the outer face. For example, every triangulation is a near-triangulation, where the outer face is also a triangle. Tutte [19] proved that every near-triangulation has a crossing-free straight-line drawing such that the vertices of the outer face are mapped to the vertices of an arbitrary convex polygon. This was extended by Hong and Nagamochi [13] to arbitrary star-shaped polygons (Lemma 2 below). A star-shaped polygon P is defined in terms of visibility. Let P be a closed polygonal domain (for short, *polygon*) bounded by a simple cycle. Two points, p and q, are mutually *visible* with respect to P if the line segment pq lies in P. The *kernel* of P, denoted $\ker(P)$, is the set of all points in P from which all vertices of P are visible. A polygon is *star-shaped* if it has a non-empty kernel.

Lemma 2 *(Hong and Nagamochi [13]) Let G be a polyhedral graph where the outer face is bounded by a cycle with t vertices $(v_1, \ldots, v_t)$. Let $(p_1, \ldots, p_t)$ be a star-shaped polygon with t vertices. Then G has a crossing-free straight-line drawing $p : V \to \mathbb{R}^2$ such that $p(v_i) = p_i$ for $i = 1, \ldots, t$.*

If T is a polyhedral graph embedded in the plane, then a simple cycle $C^*(f_1^*, \ldots, f_k^*)$ of the dual graph can be represented by a simple closed curve $\gamma = \gamma(C^*)$ that visits the corresponding faces $f_1, \ldots, f_k$ of T in this order. For an inductive argument, it is convenient to work with such a closed curve γ in an arbitrary crossing-free drawing of T.

Lemma 3 *Let $T = (V, E)$ be a 3-connected near-triangulation, and let $p : V \to \mathbb{R}^2$ be a crossing-free straight-line drawing of T such that the outer face is $(v_1, \ldots, v_t)$. Let γ be a closed Jordan curve that does not pass through any vertex of T and crosses k distinct edges $(e_1, \ldots, e_k)$ in this order, where exactly two of these edges are on the boundary of the outer face, say, $e_1 = v_1v_2$ and $e_k = v_\tau v_{\tau+1}$ with $2 \le \tau < t$. Let $P = (p_1, p_2, \ldots, p_t)$ be a star-shaped simple polygon such that there is a line L that intersects the interior of $\ker(P)$ and crosses sides p_1p_2 and $p_\tau p_{\tau+1}$ (but no other side of P).*

Then T has a crossing-free straight-line drawing $p' : V \to \mathbb{R}^2$ such that $p'(v_i) = p_i$ for $i = 1, 2, \ldots, t$, and the edges crossed by the line L are $e_1, \ldots e_k$ in this order.

Proof. We proceed by induction on k, the number of edges crossed by γ. Assume that $k \geq 3$, and Lemma 3 holds for every k' satisfying $3 \leq k' < k$.

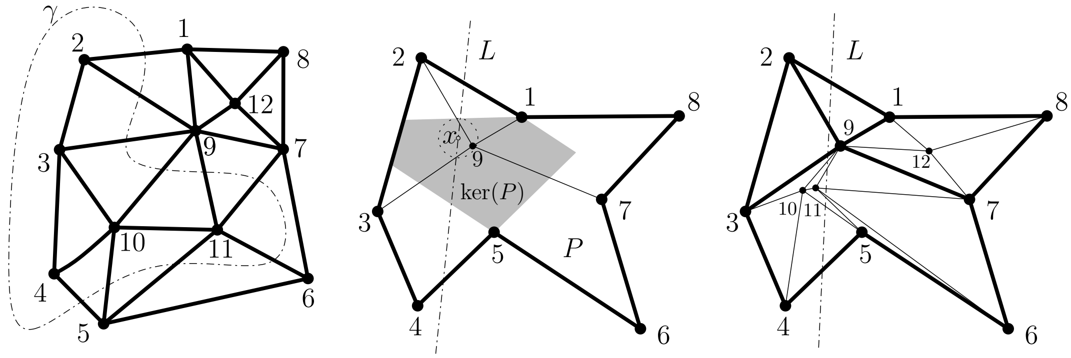

Figure 2: Left: a near-triangulation T, curve γ is a closed Jordan curve corresponding to a simple cycle in the dual graph T^*. Middle: A star-shaped polygon P with a shaded kernel $\ker(P)$. Vertex $w = v_9$ is embedded at a small neighborhood of a point $x \in L \cap \text{int}(\ker(P))$. Right: we apply induction in each bounded face of T_w.

Refer to Fig. 2. Edges $e_1 = v_1v_2$ and e_2 are two sides of a triangle f_2, and so they have a common endpoint. Assume without loss of generality that $e_2 = v_2w$, with $w \neq v_1$. Denote by T_w the subgraph of T induced by the vertex set $\{v_1, v_2, \ldots, v_t, w\}$. Since T is 3-connected, T_w consists of the chordless cycle $(v_1, v_2, \ldots, v_t)$, and a star between w and some vertices of $\{v_1, v_2, \ldots, v_t\}$ (including the edges v_1w and v_2w). All bounded faces of T_w are incident to w, and they are each bounded by chordless cycles. For every bounded face F of T_w, the subgraph $T(F)$ of T induced by the vertices on the boundary and inside F is a 3-connected near-triangulation. Indeed, suppose to the contrary that a graph $T(F)$ has a 2-vertex cut $\{a, b\}$. Since T is 3-connected, the the deletion of $\{a, b\}$ disconnects the outer face of $T(F)$. Hence both a and b are incident to the outer face and some common interior face of $T(F)$. However, the interior faces of $T(F)$ are triangles, and so ab is a chord of the outer face of $T(F)$, contradicting the fact that this outer face is a chordless cycle.

We are now ready to construct a crossing-free straight-line drawing p'. First, embed the vertices of T_w as follows. Let x be an intersection point of L and the interior of $\ker(P)$, and note that a small neighborhood of x is contained in $\ker(P)$. Let $p'(v_i) = p_i$ for $i = 1, \ldots, t$, and let $p'(w)$ be a point sufficiently close to x on the side of line L that does not contain p_2. If w is sufficiently close to x, then all bounded faces of T_w are star-shaped, and whenever L crosses a bounded face of T_w, it also intersects the kernel of that face.

Note that γ crosses exactly two edges of the outer cycle $(v_1, \ldots, v_t)$, and so it partitions the vertices $\{v_1, \ldots, v_t\}$ into two classes lying in the interior and exterior of γ, respectively. This ensures that γ and L partitions the vertices of T_w in the same two classes. Since γ crosses every edge at most once, it crosses an edge of T_w if and only if the edge connects vertices on opposite sides of γ. Analogously, L crosses an edge of T_w if and only if the edge connects vertices on opposite sides of L. Hence γ and L cross the same edges of T_w incident to w, moreover both cross these edges in the same order determined by the circular order of all edges incident to w. Consequently, γ and L also intersects the same faces of T_w, in the same order.

For each bounded face F of T_w, we can recurse on the subgraphs $T(F)$ induced by the vertices on the boundary and inside F. Specifically, if γ traverses a bounded face F of T_w, we can apply the induction hypothesis, otherwise we can embed the interior vertices of $T(F)$ using Lemma 2. □

We are now ready to prove the converse of Observation 1.

Theorem 2 *Let $T = (V, E)$ be a polyhedral graph on $n \geq 4$ vertices, $C^* = (f_1^*, \ldots, f_k^*)$ a simple cycle in the dual graph T^*, and L a line in the plane. Then T admits a crossing-free straight-line drawing such that f_1 is the outer face, and L intersects $f_1, \ldots, f_k$ in this order.*

Proof. We are given a polyhedral graph $T = (V, E)$ and a simple cycle $C^* = (f_1^*, \ldots, f_k^*)$ in the dual graph T^*. Consider an arbitrary crossing-free straight-line drawing $p : V \to \mathbb{R}^2$ of T such that the outer face is f_1. Let γ be a closed Jordan curve that corresponds to the simple cycle $C^* = (f_1^*, \ldots, f_k^*)$, that is, γ traverses faces $f_1, \ldots, f_k$ in this order in the drawing p. Augment T with dummy edges to a near-triangulation T' by triangulating all bounded faces if necessary. We may assume that γ traverses every triangular face at most once. Denote the sequence of edges of T' crossed by γ by $e_1, \ldots, e_{k'}$, where e_1 and $e_{k'}$ are adjacent to the outer face. If face f_1 has t vertices then let $P = (v_1, \ldots, v_t)$ be an arbitrary convex polygon with t vertices. By Lemma 3, T' has a crossing-free straight-line drawing such that the outer face is f_1 and a line L crosses the edges $e_1, \ldots, e_{k'}$ in this order. After deleting the dummy edges, we obtain a crossing-free straight-line drawing of T such that the outer face is f_1 and the line L stabs the faces $f_1, \ldots, f_k$ in this order, as required. □

Biedl *et al.* [1] showed that the ratio between the longest and the shortest edges in the resulting crossing-free straight-line drawing is at least $\Omega(2^{n/2})$ for some planar graphs with n vertices.

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